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# Homogenization method for elastic materials

F. Seifrt*

*Faculty of Applied Sciences, UWB in Pilsen, Univerzitní 22, 306 14 Plzeň, Czech Republic*

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**Abstract**

In the paper we study the homogenization method and its potential for research of some phenomenons connected with periodic elastic materials. This method will be applied on partial differential equations that describe the deformation of a periodic composite material. The next part of the paper will deal with applications of the homogenization method. The importance of the method will be discussed more detailed for the exploration of the so called bandgaps. Bandgap is a phenomenon which may appear during vibrations of some periodically heterogeneous materials. This phenomenon is not only observable during vibrations for the aforementioned materials, but we may also observe similar effects by propagation of electromagnetic waves of heterogeneous dielectric medias.


*Keywords:* composite materials, homogenization, finite element method, bandgaps

## 1. Introduction

The aim of this paper is to study the homogenization method applied on elastic materials. We would like to show for which problems is the method particularly important and what are the advantages that this method brings. The author was inspired by the book [2].

The two-scale method of homogenization was firstly introduced by G. Nguetseng in a year 1989 ([3]). Among other quite often cited authors belongs G. Allaire ([1]). The aim of homogenization method is to simplify description of behaviour of heterogeneous materials. Heterogeneous material is replaced by a 'homogenized', fictive material which is a good approximation of the original heterogeneous material.

## 2. Description of geometry

In the following we describe the geometry of the problem (cf. fig. 1). The macro-structure is composed of $N \times N$ cells each of size $\varepsilon$ and fills a bounded domain $\overline{\Omega}$. The domain $\Omega$ is split into domain $\Omega_1^\varepsilon$ made of elastic material 1 and domain $\Omega_2^\varepsilon$ which includes periodically distributed inclusions made of material 2. The reference cell $Y = [0,1[^3$ is composed from the elementary inclusion $Y_2$, $\overline{Y_2} \subset Y$ and the matrix $Y_1 = Y \setminus \overline{Y_2}$, hence we have

$$\Omega_2^\varepsilon = \bigcup_{k \in \mathbb{K}^\varepsilon} \varepsilon(\overline{Y_2} + k), \quad \mathbb{K}^\varepsilon = \{k \in \mathbb{Z}^3, \varepsilon(\overline{Y_2} + k) \subset \Omega\}, \tag{1}$$

$$\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}. \tag{2}$$

The size of the whole domain is fixed. What changes with $\varepsilon$ is the size of periodically repeating cells.

*Corresponding author. Tel.: +420 377 632 365, e-mail: seifrt@kme.zcu.cz.

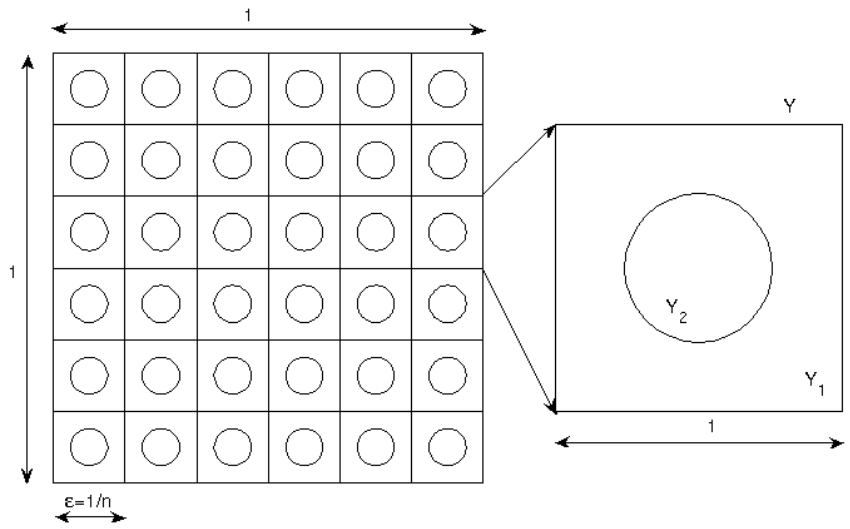


Fig. 1. Description of geometry.

Let us now establish the macro and micro coordinates system. The couple $(x_1, x_2)$ represents macro coordinates system which corresponds with the position in the structure of $N \times N$ cells. The couple $(y_1, y_2)$ matches with the micro coordinates system and provide position within the reference cell $Y$.

## 3. State equations

The state equations will in our case describe the deflection of a loaded board. Let $c_{ijkh}(\boldsymbol{y})$ be material coefficients of the reference cell. Then we see that the function[1]

$$c^{\varepsilon}_{ijkh}(\boldsymbol{x}) = c_{ijkh}\left(\frac{\boldsymbol{x}}{\varepsilon}\right) \tag{3}$$

is periodical and corresponds with the material coefficients of a board composed from $N \times N$ cells. The material coefficients $c^{\varepsilon}_{ijkh}(\boldsymbol{x})$ are defined on $\Omega$ and for smaller $\varepsilon$ provide finer and finer periodic structure. The board is loaded with a force $\boldsymbol{f}$. The board responds with the deflection $\boldsymbol{u}^{\varepsilon}$. The situation is described by the following equations in the classical sense[2]

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(c^{\varepsilon}_{ijkh}(\boldsymbol{x})\dfrac{\partial u^{\varepsilon}_k}{\partial x_h}\right) = f_i & \text{v} \quad \Omega, \\ \boldsymbol{u}^{\varepsilon}(\boldsymbol{x}) = \boldsymbol{0} & \text{na} \quad \partial\Omega \end{cases} \tag{4}$$

and corresponding variational form

$$\begin{cases} \text{Find } \boldsymbol{u}^{\varepsilon} \in \boldsymbol{H}^1_0(\Omega) \text{ such that} \\ \displaystyle\int_{\Omega} c^{\varepsilon}_{mnkl} e_{kl}(\boldsymbol{u}^{\varepsilon}) e_{mn}(\boldsymbol{\Phi}) = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{\Phi} \quad \forall \boldsymbol{\Phi} \in \boldsymbol{H}^1_0(\Omega), \end{cases} \tag{5}$$

[1]As foresaid $\boldsymbol{x} = (x_1, x_2)$ are global coordinates and $\boldsymbol{y} = (y_1, y_2) = \dfrac{\boldsymbol{x}}{\varepsilon}$ are the local coordinates on the reference cell.

[2]Latin exponents and indices take their values in the set $\{1, 2\}$. Einstein convention for repeated exponents and indices is used. Bold face letters represent vectors or vector spaces.

where under $e_{kl}$ we mean the Cauchy tensor of small deformations

$$e_{kl}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right) \tag{6}$$

and the space $\boldsymbol{H}_0^1(\Omega)$ is the Sobolev space $\boldsymbol{H}^1(\Omega)$ with compact support from which we take the state and test functions.

Now we get closer to the equations of homogenized fictive material. Material coefficients for heterogeneous material $c^{\varepsilon}_{ijkh}$ in the equation (4) will be replaced by homogeneous coefficients $c^{*}_{ijkh}$, the state equation will be then rewritten as follows

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(c^{*}_{ijkh}(\boldsymbol{x})\dfrac{\partial u_k}{\partial x_h}\right) = f_i & \text{in} \quad \Omega, \\ \boldsymbol{u}^{\varepsilon}(\boldsymbol{x}) = \boldsymbol{0} & \text{on} \quad \partial\Omega. \end{cases} \tag{7}$$

Homogeneous coefficients (sometimes also called effective parameters) equal to the difference of integral average of heterogeneous material coefficients over the domain of the reference cell $Y$ and corrector coefficients

$$c^{*}_{ijkh} = c^{\text{average}}_{ijkh} - c^{\text{corrector}}_{ijkh}. \tag{8}$$

Integral average of heterogeneous material coefficients is defined by the following relation

$$c^{\text{average}}_{ijkh} = \frac{1}{|Y|}\int_Y c_{ijkh}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{9}$$

Corrector coefficients may be obtained as integral average for which we have to introduce auxiliary periodic functions $\boldsymbol{\chi}^{kh}$

$$c^{\text{corrector}}_{ijkh} = \frac{1}{|Y|}\int_Y c_{ijlm}(\boldsymbol{y})\frac{\partial \chi^{kh}_l}{\partial y_m}\,\mathrm{d}\boldsymbol{y}. \tag{10}$$

Auxiliary functions $\boldsymbol{\chi}^{kh}$ are the solution of the following variational formulas

$$\int_Y c^{\varepsilon}_{ijkh} e_{ij}(\boldsymbol{\chi}^{ij}) e_{kh}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{y} = \int_Y c^{\varepsilon}_{lmkh} e_{kh}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{y} \quad \forall \boldsymbol{v} \in \boldsymbol{W}^1_{per}(Y), \tag{11}$$

where $\boldsymbol{W}^1_{per}(Y)$ is the space of Y-periodic functions with a zero integral average

$$\boldsymbol{W}^1_{per}(Y) = \left\{\boldsymbol{v} \middle| \boldsymbol{v} \in \boldsymbol{H}^1(Y), \int_Y v_i\,\mathrm{d}\boldsymbol{y} = 0, i = 1,2\right\}. \tag{12}$$

## 4. Discretization

Discretization of the state equations was done by the classical approach of the finite element method (for details we recommend the well known book [6]). We used linear finite elements on a triangular mesh with isoparametric representation. The problem of linear elasticity (5) may be formulated as follows

$$\boldsymbol{K}^{\varepsilon}\boldsymbol{u}^{\varepsilon} = \boldsymbol{f}, \tag{13}$$

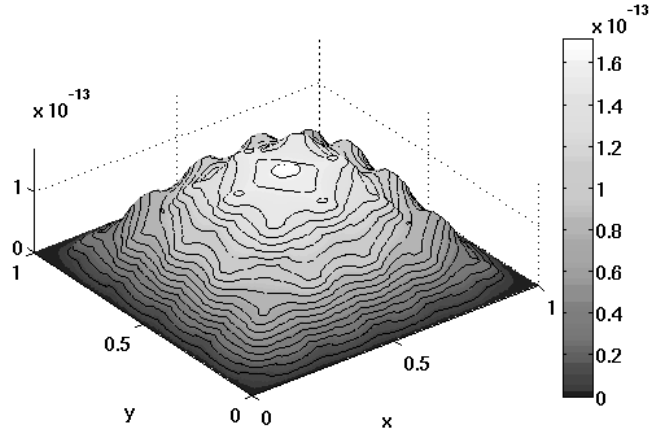


Fig. 2. Magnitude value of the displacement $\boldsymbol{u}^\varepsilon$ corresponding to the heterogeneous material.

where $\boldsymbol{K}^\varepsilon$ is the global stiffness matrix defined as

$$\boldsymbol{K}^\varepsilon = \sum_e \boldsymbol{K}_e^\varepsilon, \tag{14}$$

and the right hand side corresponds to the force

$$\boldsymbol{f}^\varepsilon = \sum_e \boldsymbol{f}_e^\varepsilon. \tag{15}$$

The element stiffness matrix $\boldsymbol{K}_e^\varepsilon$ depends on heterogeneous material coefficients $c_{ijkh}^\varepsilon(\boldsymbol{x})$. Similarly we may formulate the discrete version of the state equation (7) for the fictive homogenized material

$$\boldsymbol{K}^* \boldsymbol{u} = \boldsymbol{f}. \tag{16}$$

## 5. Numerical results

In our example we consider only a 2D case - plane deformation. The board is composed from $5 \times 5$ cells. Each cell is a composition of epoxy inclusion and aluminium matrix. The force acts on the whole domain in the direction of the axis of the first quadrant in the $x - y$ plane. On the figures 2 and 3 we display the magnitude of the displacements $\boldsymbol{u}^\varepsilon$ ($\boldsymbol{u}$) of heterogeneous (homogeneous) material. By the magnitude values we mean the euclidean norms of displacements evaluated at each node. On the last figure 4 you may see the $L^2$ norm of displacements for different values of $\varepsilon$. We observe that for $\varepsilon < 1/2$ the norms of displacements $\boldsymbol{u}^\varepsilon, \boldsymbol{u}$ are relatively near. Convergence analysis is beyond this text (again we refer to [2]).

On this place we would like to emphasize the main advantage of the homogenization method. This approach reduces computational costs very significantly. In the case of heterogeneous material we have to use a relatively fine mesh to catch details on interfaces between materials. We don't have to take care of this problem in the case of the fictive material. And therefore tasks of greater complexity may be solved.

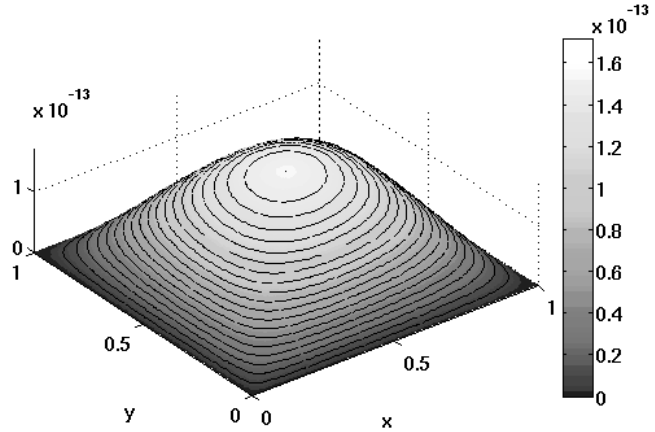


Fig. 3. Magnitude value of the displacement $\boldsymbol{u}$ corresponding to the fictive material.

On the other side the range of problems on which the method of homogenization may be applied is limited and in particular cases one should always answer the question how good approximation the fictive material really is.

## 6. Bandgaps

Recently it has been shown that heterogeneous elastic material with a periodic structure can exhibit acoustic band gaps (cf. [5]). Phononic band gaps are certain frequency ranges for which elastic or acoustic waves cannot propagate. Therefore these materials (often called phononic crystals) could be used as frequency filters, vibration dampers or waveguides. One possible approach for studying the phenomena mentioned above are homogenization techniques.

The state equation for amplitudes of elastic waves is formulated similarly as the previous state equation for the linear elasticity (5)

$$\omega^2 \int_\Omega r^\varepsilon \boldsymbol{u}^\varepsilon \cdot \boldsymbol{\Phi} - \int_\Omega c^\varepsilon_{mnkl} e_{kl}(\boldsymbol{u}^\varepsilon) e_{mn}(\boldsymbol{\Phi}) = - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\Phi} \quad \forall \boldsymbol{\Phi} \in \boldsymbol{H}^1_0(\Omega), \tag{17}$$

where $r^\varepsilon$ is the mass density and $\omega$ is a given angular frequency of the incident elastic wave. The reader surely noticed that according to the previous example we have to manage with one additional term coming from the inertial force. For the zero angular frequency we get exactly (5)[3]. For $\omega$ different from the resonance values the variational problem has a unique solution $\boldsymbol{u}^\varepsilon \in \boldsymbol{H}^1_0(\Omega)$.

In the discrete case additionally to the stiffness matrix $\boldsymbol{K}$ and the right hand side vector of forces $\boldsymbol{f}$ we obtain the mass matrix $\boldsymbol{M}$.

$$(\boldsymbol{K}^\varepsilon - \omega^2 \boldsymbol{M}) \boldsymbol{u}^\varepsilon = \boldsymbol{f}. \tag{18}$$

Now the method of homogenization provide us the homogenized mass and stiffness matrix and the homogenized right hand side. Studying of the homogenized mass matrix provide as valuable information about the bandgaps. Detailed report on this topic may be found in [4].

[3]Except that in this case we assumed a different elastic material with a strong heterogeneity - relation (3) is more complex (see [4])

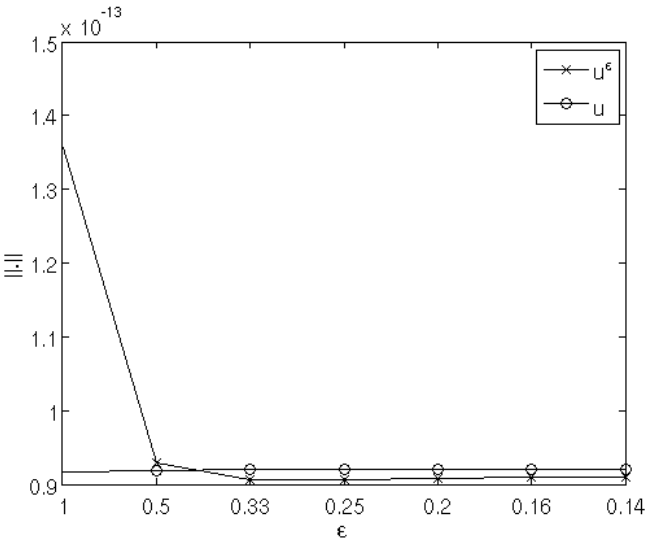


Fig. 4. $L^2$ norm of displacements $\boldsymbol{u}^\varepsilon, \boldsymbol{u}$.

## 7. Conclusion

We have demonstrated an idea of homogenization. Further we have mentioned some advantages and disadvantages of this approach. The method has been applied on a linear elastic material which is a composition of epoxy and aluminium. We saw that the resulting fictive material is a relatively good approximation of the original heterogeneous material. Since the reduce of computational cost by the use of the homogenization method is essential it's reasonable to continue in this direction. The author's future plan is to study the method of homogenization and perhaps apply it in the case of electromagnetic waves.

## Acknowledgement

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