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Application of stabilization techniques in the dynamic analysis of multibody systems

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Abstract

This paper is intended to the discussion of possible methods for the solution of the motion equations of constrained multibody systems. They can be formulated in the form of differential-algebraic equations and their numerical solution brings the problems of constraint violation and numerical stability. Therefore special methods were proposed to handle these problems. Various approaches for the numerical solution of equations are briefly reviewed and the application of the Baumgarte's stabilization method on testing examples is shown. The paper was motivated by the effort to find the suitable solution methods for the equations of motion in the form of differential-algebraic equations using the MATLAB standard computational system.

Keywords: dynamics, multibody systems, differential-algebraic equations, numerical simulation

1. Introduction

Multibody systems are common mechanical structures known from various branches of engineering. Since the second half of the 20th century the methods of the multibody system dynamics have been extensively developing. This expansion is possible mainly due to the development of computational hardware and increasing power of computers, because the resulting mathematical models are mostly the sets of nonlinear differential and algebraic equations. Not only their solution but also the creation of these complex sets of equations for large multibody systems cannot be performed manually and thus the help of computers is necessary.

There is plenty of software tools specialized in the field of multibody dynamics, which can be currently used for the solution of the engineering or research problems (ADAMS, SIMPACK, *alaska*, …). The equations of motion while working with these tools are created on the basis of so called multibody formalisms (see e.g. [21]). They are special proposed algorithms for the automatic generation of equations of motion of the coupled rigid body systems. Also flexible bodies can be incorporated in computational models. The software tools integrate more or less comfortable pre- and post-processing environment with efficient solvers developed by groups of mathematicians and engineers. On the other hand it is sometimes more advantageous to formulate and to solve the equations of motion of a studied multibody system without usage of the commercial software tools. The main reason is the black-box-like behaviour of the commercial tools and the limited possibility of introducing some special features and special model elements as well as some non-standard solution or optimization methods.

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This paper is intended for the discussion of possible methods for the solution of the motion equations of multibody systems. The approaches for the formulation and numerical solution of equations are briefly described. The Baumgarte's stabilization method is used together with the chosen integration methods for the numerical simulation of the testing examples. The paper was motivated by the effort to find the suitable solution methods for the equations of motion in the form of differential-algebraic equations using the MATLAB standard computational system.

2. Equations of motion of multibody systems

One of the most important things in the formulating of multibody equations of motion is the proper selection of the type of coordinates describing the system of coupled rigid bodies. The position of each rigid body in the 3D space can be determined for example by six coordinates (three displacements and three rotation). If the bodies are coupled together by joints those body coordinates are constrained. The review of different coordinate types can be found in many textbooks, e.g. [6], [16], [20], [21].

The above mentioned coordinates describing the position of each body in the space (2D or 3D) are called Cartesian, physical or reference point coordinates because they determine the position of the reference point (mostly centre of mass) and the rotation of the local body coordinate system. Relative coordinates define the position of particular bodies with respect to the previous body in the kinematic chain. Natural coordinates determine the position of the body by means of the Cartesian coordinates of the basic points and by means of the components of some unit vector [6]. As another coordinate type can be also denoted independent generalized coordinates that are connected with the degrees of freedom of the multibody system. The names of the coordinate types may vary according to the authors of the books but the meaning is the same.

Formulation of equations of motion using any of these coordinate types except independent generalized coordinates leads to the mathematical model in the form of a set of differential-algebraic equations (DAEs). Various approaches for the creation of the motion equations of the multibody systems can be found in most monographs specialized in the multibody dynamics ([1], [6], [20], [21] etc.). The techniques based on the vector mechanics as the direct application of Newton's second law in connection with the free body principle can be used but it becomes difficult when large-scale multibody systems are investigated. Suitable methods for the formulation of the motion equations of more general and complex multibody systems are based on analytical mechanics. They are of various complexity. One of the basic ones is the principle of virtual work. Other principles as Jourdain's or Hamilton's and other methods can also be employed. Their advantage is in the formulation using scalar mechanical quantities such as kinetic and potential energy characterizing a multibody system. The most often used are the Lagrange's equations (of the mixed type for dependent generalized coordinates $\boldsymbol{q}$, [20], [21])

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_k}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial E_k}{\partial \boldsymbol{q}} = \boldsymbol{Q} - \frac{\partial E_p}{\partial \boldsymbol{q}} - \frac{\partial R}{\partial \dot{\boldsymbol{q}}} + \boldsymbol{\Phi}_q^T \boldsymbol{\lambda}, \tag{1}$$

where E_k is the kinetic energy, E_p is the potential energy, R is the Rayleigh's dissipation function, $\boldsymbol{Q}$ is the vector of generalized forces corresponding to particular generalized coordinates. Holonomic rheonomic constraints between the chosen coordinates can be written using the vector notation

$$\boldsymbol{\Phi}(\boldsymbol{q},\, t) = \mathbf{0} \tag{2}$$

and for the use in equation (1) it must be differentiated to obtain the Jacobian matrix

$$\boldsymbol{\Phi}_q = \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{q}} = \left[\frac{\partial \Phi_i}{\partial q_j} \right], \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \tag{3}$$

where m is the number of constraints and n is the number of dependent generalized coordinates. Further vector of Lagrange multipliers $\boldsymbol{\lambda}$ is introduced in equation (1).

After the substitution of the particular expressions for the energies and for the generalized forces the equations of motion

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) - \boldsymbol{\Phi}_q^T \boldsymbol{\lambda} = \boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \tag{4}$$

together with the constraint equations (2) constitute the mathematical model of the constrained multibody system. Matrix $\boldsymbol{M}$ is the global mass matrix of the multibody system and vector $\boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ contains centrifugal and Coriolis inertia forces, elastic and damping forces and other externally applied forces including a gravity.

An important characteristic of the differential-algebraic equations is their differential index (often referred to as index only). It can be defined [6] as the number of times that the DAE has to be differentiated to obtain a standard set of ordinary differential equations (ODEs). Another more formal definition can be found e.g. in [19]. The higher the index the more complex and difficult the integration of DAEs. It can be easy shown that the mathematical model (2) and (4) is the index three DAE. Therefore from the viewpoint of the solution methods the mathematical model is usually transformed into the set of differential-algebraic equations of index one

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{\Phi}_q^T \\ \boldsymbol{\Phi}_q & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \\ \gamma(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \end{bmatrix} \tag{5}$$

by the double differentiation of the constraint equations with respect to time. Vector $\boldsymbol{\gamma}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ represents the remaining terms after the constraints differentiation.

Another classification of differential equations can divide the ODEs to the non-stiff and stiff systems. It holds simply for the stiff system that its eigenfrequencies are distributed over a broad frequency range. This fact brings difficulties in numerical integration and therefore special care has to be taken on this property.

3. Numerical solution of the mathematical model

If the mathematical model of the multibody system is formulated in one of the above introduced forms the main problem is the selection of the proper solution method.

It is convenient to avoid the solution of the index three formulation (2) and (4) because of the difficulties associated with the solution of these equations. The simplest way is to solve directly the index one formulation (5) by rewriting this equation to the expression of acceleration vector $\ddot{\boldsymbol{q}}$ and consecutive application of some scheme for the direct numerical integration. These algorithms are described e.g. in [6], [16].

The similar approach is based on the transformation of the index one DAE into the so called underlying ODE by the elimination of Lagrange multipliers. To avoid the computation of the Lagrange multipliers, the accelerations

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1} \left(\boldsymbol{g} + \boldsymbol{\Phi}_q^T \boldsymbol{\lambda} \right) \tag{6}$$

can be expressed from the first equation of (5). Introducing this form of the acceleration vector in the second equation of (5) one can get

$$\boldsymbol{\Phi}_q \boldsymbol{M}^{-1} \left(\boldsymbol{g} + \boldsymbol{\Phi}_q^T \boldsymbol{\lambda}\right) = \boldsymbol{\gamma}. \tag{7}$$

After rearranging this expression Lagrange multipliers can be expressed in the form

$$\boldsymbol{\lambda} = \left(\boldsymbol{\Phi}_q \boldsymbol{M}^{-1} \boldsymbol{\Phi}_q^T\right)^{-1} \left(\boldsymbol{\gamma} - \boldsymbol{\Phi}_q \boldsymbol{M}^{-1} \boldsymbol{g}\right). \tag{8}$$

Finally the vector $\boldsymbol{\lambda}$ can be eliminated and thus

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1} \boldsymbol{\Phi}_q^T \left(\boldsymbol{\Phi}_q \boldsymbol{M}^{-1} \boldsymbol{\Phi}_q^T\right)^{-1} \left(\boldsymbol{\gamma} - \boldsymbol{\Phi}_q \boldsymbol{M}^{-1} \boldsymbol{g}\right) + \boldsymbol{M}^{-1} \boldsymbol{g}. \tag{9}$$

This equation can be solved using the standard numerical integration methods but it has some undesirable properties. It can be numerically unstable for certain cases and the so called drift-off effect consisted of the violation of constraints during the solution occurs. It is influenced by round-off errors and discretization due to the numerical approximation. The violation problem can be seen from the fact that numerical solution using the second derivative of the constraints is not the same as the solution using the original constraints equations. Therefore various techniques were studied to improve the solution of the multibody equations of motion in the index one or index three formulation and many algorithms were proposed.

A comprehensive general review of them is presented in [3] and [14]. Less substantial survey of these methods can be found in [5], [6], [18] and [21]. The methods for the numerical integration of ODEs in dynamics are reviewed e.g. in [4], [7], [6], [16]. The approaches to the treatment of the constraint violation and bad stability can be viewed very roughly as two big groups. The solution can be stabilized during the integration process or the constraints can be reduced and thus eliminated from the process.

3.1. Stabilization techniques

One type of the methods is called a constraint violation stabilization [3], [20] or a constraint regularization [9] with the well known Baumgarte's stabilization method [1], [6], [16], [21]. The constraint equations are modified and equation

$$\ddot{\boldsymbol{\Phi}} + 2\alpha\dot{\boldsymbol{\Phi}} + \beta^2 \boldsymbol{\Phi} = \boldsymbol{0} \tag{10}$$

is solved instead of the equation $\ddot{\boldsymbol{\Phi}} = \boldsymbol{0}$ during the numerical solution of the index one DAE (5). Coefficients α and β are the chosen constants. Vector $\boldsymbol{\gamma}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ in (5) is then replaced by new vector

$$\bar{\gamma}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \boldsymbol{\gamma}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - 2\alpha\dot{\boldsymbol{\Phi}} - \beta^2 \boldsymbol{\Phi}. \tag{11}$$

The Baumgarte's stabilization method is one of the oldest and its origin is in the control theory. Additional terms can be explained as a PD controller [21] and they introduce feedback in the equations.

Disadvantages of the method are in the selection procedure of the coefficients α and β. Any general procedure for their proper selection has not been found. The selection of α and β such that $\beta = \alpha\sqrt{2}$ or $\beta = \sqrt{2\alpha}$ is suggested in [21]. The choice of α and β between 1 and 10 and the case of $\beta = \alpha$ to achieve critical damping is proposed in [16]. An automatic selection of these parameters using the stepsize of the numerical integration is mentioned in [22] and the stability of the method is investigated by the authors of [13]. The advantage of the Baumgarte's stabilization method is especially its easy implementation and usage.

Other stabilization techniques are based on the penalty formulation. The review of these techniques can be found in [3]. The idea of this approach is the constraints enforcing using the special penalty terms added to the Lagrangian of the system. The additional term can be e.g. $\frac{1}{2}\boldsymbol{\Phi}^T\boldsymbol{P}\boldsymbol{\Phi}$, where the matrix $\boldsymbol{P} = \mathrm{diag}(p_i^2)$ is composed of the particular penalty factors p_i. In ideal case if the penalty goes to the infinity the constraints should go to the zero. In practical cases the finite values of the penalty factors are selected. The representant of the penalty based techniques intended for the constraint stabilization is the augmented Lagrangian formulation described e.g. in [3] or [6].

The special issue of the stabilization of the motion equations of multibody systems with singular positions and redundant constraints is dealt with in [1] using a proposed Amirouche-Ider stabilization method and in [15].

3.2. Other approaches

There exist many other approaches suitable for the solution of equations of motion of the constrained multibody systems. One important group of methods are constraint reduction methods. They are based on the numerical determination of a minimal set of equations and on the solution of this set. The well-known coordinate partitioning method [11], [16], [20] can be used for this purpose. This approach uses the partitioning of the coordinates of the system into the independent and dependent ones. Only the minimal ODE set of equations is then used for the numerical integration and dependent coordinates are calculated using the relations obtained from the coordinate partitioning. Other possibilities for the constraint elimination are various projection techniques [6].

The projection methods can also be employed during the integration process. Approximately during past fifteen years different DAE solvers have been designed for the solution of nonlinear sets of differential-algebraic equations and the DAEs theory is still studied. The adaptive Adams integration method in connection with the projection on the constraint manifold in each integration step is proposed in [19]. The variable step size strategy is also discussed there. The linear implicit Euler method and the projection technique are used in [5] for the real-time simulations of multibody systems. The Newmark method is used in [8] for the solution of the index three DAE. The stability of this method intended for the DAEs is studied in [9]. Other applications of the classical methods for direct integration of differential-algebraic equations are described in [6] (BDF, Runge-Kutta) and in [17] (Adams-Bashfort-Moulton).

The simple direct violation correction method is shown in [22]. The implicit constraint enforcement scheme is proposed by the authors of [12]. Also new developed energy conserving algorithms (see e.g. [3], [9]) known mainly from the structural dynamics are utilizable in multibody dynamics.

4. Testing problems

As it was mentioned in the introduction, this paper is motivated by the effort to find the suitable solution method for the dynamics of the constrained multibody systems that can be easily implemented in some general computational environment. The MATLAB system offers a number of the methods for direct numerical integration of ODEs [2] and it can be used for the implementation of the chosen approaches for the constraint stabilization or reduction. Due to its simplicity the Baumgarte's stabilization method was chosen for the constraint violation improvement in this paper. Two multibody systems were used to test the suitability and efficiency of the combination of the MATLAB ODE suite with the Baumgarte's stabilization method for the solution of equations of motion.

4.1. Four-bar mechanism

The four-bar mechanism (fig. 1) was chosen as the example of the simple mechanical system. The particular lengths was chosen as $l_1 = 0.2$ m, $l_2 = 0.4$ m, $l_3 = 0.25$ m, $d = 0.4$ m. The gravity forces acting on the bars and driving torque M were considered. The multibody system consists of three bodies and therefore the number of coordinates is 9. The number of constraint equations is 8.

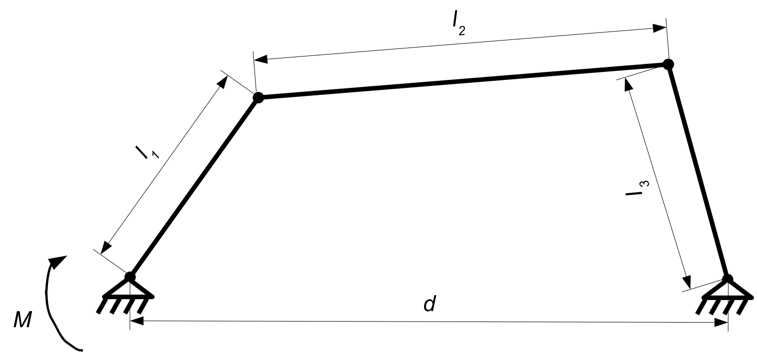

Fig. 1. Scheme of the four bar mechanism.

The `ode23` function with a default error setting based on the Runge-Kutta method was used for the numerical integration of the equations of motion in MATLAB. Other functions and different error tolerances were also tested but the general conclusions about the stabilization efficiency are the same. The time history of the chosen constraint equation fulfillment for correct initial conditions is shown in fig. 2. When the false initial conditions are prescribed, the stabilization method is able to enforce the correct constraints and it leads to their fulfillment (see fig. 3).

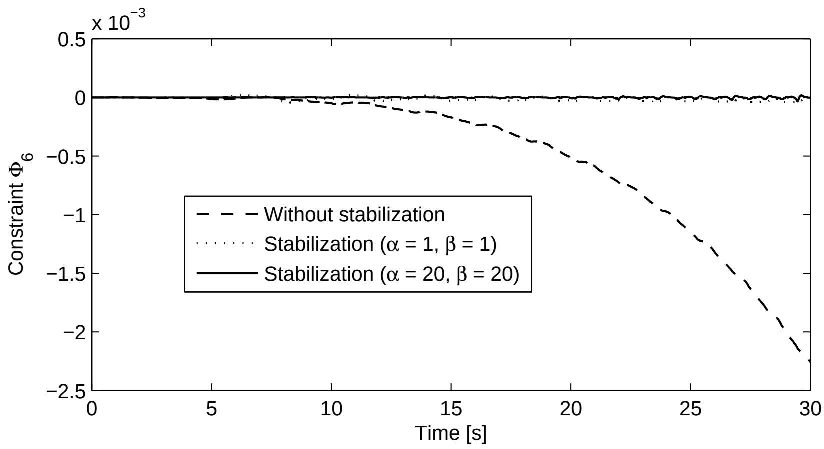

Fig. 2. Fulfillment of the chosen constraint equation for various parameters of the stabilization (simulation of the four-bar mechanism).

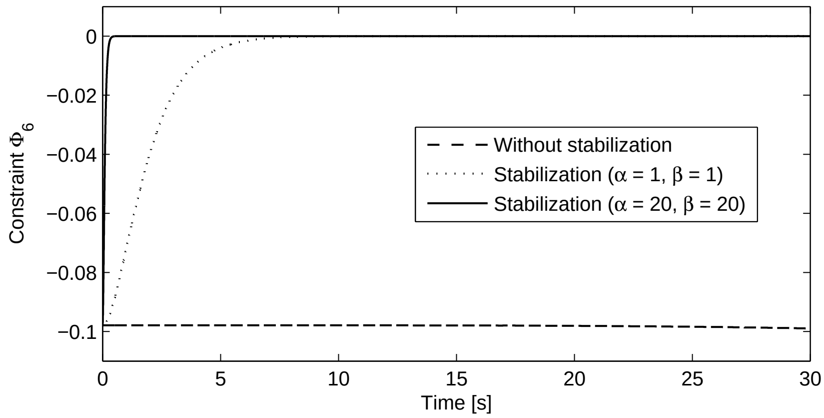

Fig. 3. Fulfillment of the chosen constraint equation for various parameters of the stabilization (simulation of the four-bar mechanism with incorrect initial conditions).

4.2. Falling flexible rod with possible contacts

The problem of the flexible rod falling in fluid considering possible contacts with adjacent non-moving parts is motivated by the tasks from the nuclear engineering. The multibody approaches were employed [10] for the dynamic analysis of the complex models of the control assemblies in three types of nuclear reactors. In order to obtain deeper knowledge about the system behaviour the problem was simplified and one flexible rod was considered.

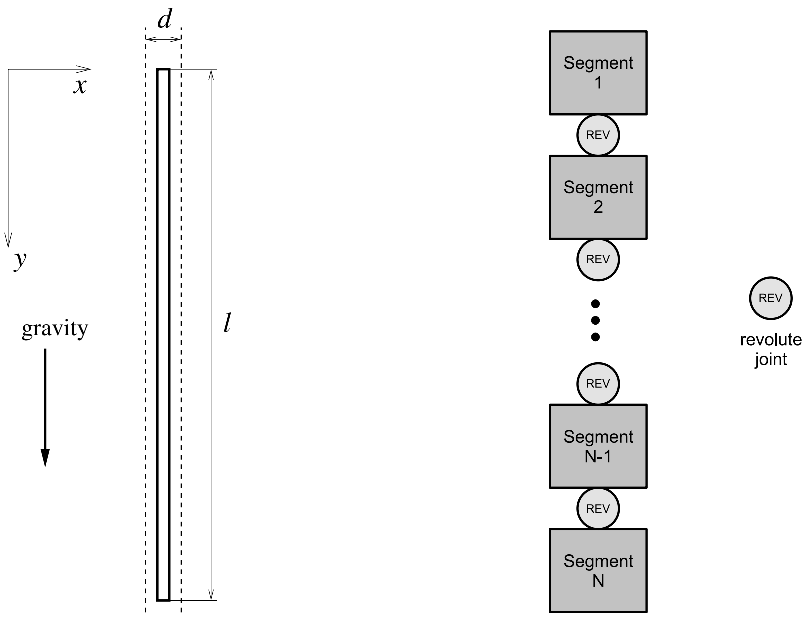

Fig. 4. Scheme of the test rod with global coordinate system and its kinematic scheme.

The scheme of the test rod is shown in fig. 4. Its length is $l = 1$ m and width is $w = 0.03$ m. The rod is falling in the channel (tube) of width $d = 0.05$ m, which is filled with water like in the nuclear reactor. The task is considered a planar problem for the sake of simplicity. The particular values of parameters were chosen for testing purposes only. The flexibility of the rod is introduced by the finite segment method, i.e. the whole rod is articulated into the set of rigid bodies (N segments of lengths l_i, $N = 10$) coupled by revolute joints (see fig. 4). The torsional springs representing the bending stiffness of the rod are applied in the joints. The mass properties and stiffnesses can be determined using basic expressions. Each body i is described in a plane by Cartesian (reference point) coordinates

$$\boldsymbol{q}_i = [\, x_i \; y_i \; \varphi_i \,]^T \tag{12}$$

representing two translations of the centre of mass and rotation of the local coordinate frame with the origin in the centre of mass. Then the mass matrix of the body is

$$\boldsymbol{M}_i = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & I_i \end{bmatrix}, \tag{13}$$

where m_i is mass and I_i is inertia moment.

The hydrodynamic resistance forces in dependence of the body velocity are included for both axial and transversal directions. The influence of the water is represented also by the static uplift pressure. The possible contacts are modelled using the impact force based on the Hertzian law and the corresponding friction force. Bending stiffnesses are considered by means of the torques depending on the relative rotation between the coupled segments. All the applied generalized forces are used for the formulation of vector $\boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ on the right-hand side of the equations of motion. For the forces representation in more detail see e.g. [10].

The revolute joint between the i-th and the $(i+1)$-th segments can be formulated in the form

$$\boldsymbol{\Phi}_i = \begin{bmatrix} x_i - \frac{l_i}{2}\sin\varphi_i - x_{i+1} - \frac{l_{i+1}}{2}\sin\varphi_{i+1} \\ y_i + \frac{l_i}{2}\cos\varphi_i - y_{i+1} + \frac{l_{i+1}}{2}\cos\varphi_{i+1} \end{bmatrix} = \boldsymbol{0}. \tag{14}$$

The total number of the constraint equations is $2(N-1)$.

The initial rotation of the rod by the angle $0.6°$ and the initial velocity 0.1 m·s^{-1} of the rod in x-direction were prescribed. For the illustration the chosen results of the dynamic response are shown in fig. 5 and fig. 6. The contacts of the rod ends with the tube can be identified from these results. The effect of the stabilization is shown in fig. 7. The drift of the constraint is growing without the stabilization.

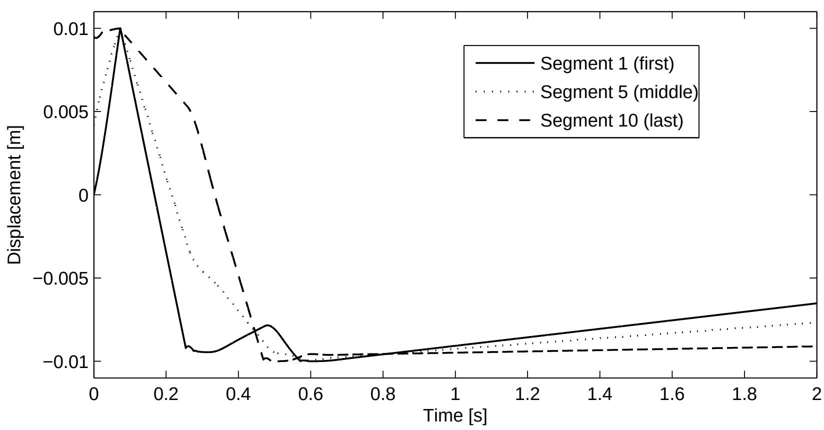

Fig. 5. Time history of the displacement of the chosen segments in x-direction (simulation of the falling rod).

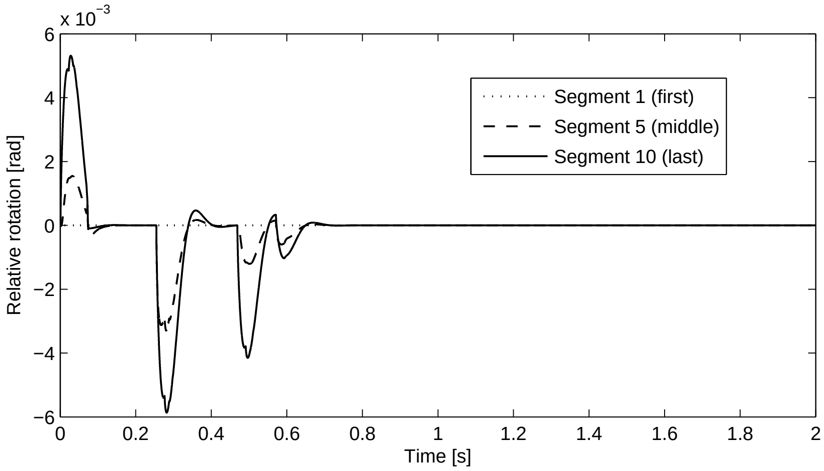

Fig. 6. Time history of the relative rotation of the chosen segments with respect to the rotation of the first segment (simulation of the falling rod).

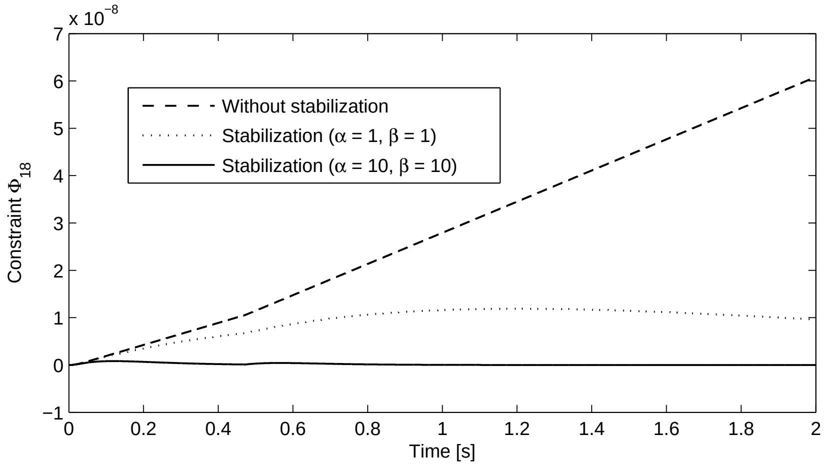

Fig. 7. Fulfillment of the chosen constraint equation for various parameters of the stabilization (simulation of the falling rod).

5. Conclusion

The paper briefly reviews the possibilities of the numerical solution of the motion equations of multibody systems. The form of equations as the sets of DAEs in the case of utilizing dependent coordinates brings the problems of the constraint violation. The Baumgarte's stabilization technique was used for the constraint stabilization. The connection of this stabilization method with the standard MATLAB's functions for the numerical integration of ODEs were approved. The chosen results of the numerical simulation of the two multibody systems illustrate the efficiency of the solution methodology.

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