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On the high order FV schemes for compressible flows

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Abstract

This work describes a high order finite volume scheme for compressible flows. The differences between the basic low order scheme and the second and third order schemes are documented for the case of inviscid flows in a channel as well as for the case of viscous flows over a profile.
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1. Introduction

This article describes the development of a high order numerical method for the solution of compressible transonic flows. The discretisation in space is based on the standard finite volume method of Godunov's type. A higher order of accuracy is achieved by the piecewise polynomial interpolation similar to the ENO or weighted ENO method (see e.g. [7]).

The here mentioned method is developed with the aim to simplify the implementation of the reconstruction procedure especially for the case of unstructured meshes. The reconstruction procedure uses single stencil and computes the interpolation polynomial by minimizing the weighted interpolation error over the cells in this stencil.

The complete finite volume scheme equipped with the piecewise linear reconstruction has been successfully used for the solution many transonic flow problems (see e.g. [3, 4]). This article presents the extension of the method to piecewise parabolic case with the application for the solution of flows through turbine cascades.

The flow is described by the set of Euler or Navier-Stokes equations in conservative form

$$W_t + F(W)_x + G(W)_y = F^v(W)_x + G^v(W)_y + S(W), \tag{1}$$

where $W = [\rho, \rho u, \rho v, e]^T$ is the vector of conservative variables, $F(W)$ and $G(W)$ are the inviscid fluxes, $F^v(W)$ and $G^v(W)$ are the viscous fluxes ($F^v = G^v = 0$ for the case of Euler equations) and $S(W)$ is a source term [2].

The equations equipped with proper boundary conditions are solved numerically using an unstructured mesh and a finite volume scheme with all unknowns located at cell centers. The inviscid fluxes through the cell interfaces are approximated by the Gauss quadrature with the physical fluxes replaced by the numerical ones

$$\int_{C_i \cap C_j} (F(W), G(W)) \cdot d\vec{S} \approx \sum_{q=1}^{J} \omega_q F^{AUSMPW+}(W^L_{ijq}, W^R_{ijq}, \vec{S_{ijq}}). \tag{2}$$

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Here $W_{ijq}^{L/R}$ denotes the values of the vector of unknowns interpolated to the Gauss point q of the interface $C_i \cap C_j$ from the left cell (superscript L) or from the right cell (superscript R). The resulting finite volume scheme for inviscid case can be then written in semi-discrete form as

$$|C_i|\frac{dW_i}{dt} = -\sum_{j\in\mathcal{N}_i}\sum_{q=1}^{J}\omega_q F^{AUSMPW+}(W_{ijq}^L, W_{ijq}^R, \vec{S_{ijq}}). \tag{3}$$

Here C_i is the i-th cell, C_i is the volume (surface in 2D) of C_i, $W_i = \int_{C_i} W(\vec{x},t)d\vec{x}$, and $\mathcal{N}_i = \{j : \dim(C_i \cap C_j) = 1\}$.

The basic first order scheme can be obtained by setting $J = 1$, $W_{ijq}^L = W_i$, and $W_{ijq}^R = W_j$.

2. The WLSQR interpolation

However the basic first order scheme posses very good mathematical properties, it is well known, that it is very diffusive. Therefore people prefer to use higher order schemes, especially for the viscous flow calculations. The higher order scheme can be constructed within this framework simply by improving the interpolation of $W^{L/R}$. There exists several methods for the construction of a stable interpolation, the most known are the limited least squares of Barth [1], the ENO/WENO schemes [7], or the TVD schemes [6].

The use of limiters as in TVD or Barth's scheme usually cut the order of accuracy near extrema and may also hamper the convergence to steady state. On the other hand, the implementation of ENO/WENO schemes is relatively complicated for unstructured meshes. Therefore a novel reconstruction procedure has been introduced in [4].

The interpolation polynomial P_i is obtained by the least square method. It is well known, that the simple least square approach fails in the case of discontinuous data (e.g. shock waves). In order to avoid the interpolation across the discontinuity, the data-dependent weights (or smoothness indicators) are introduced and the interpolation polynomial $P_i(\vec{x};\phi)$ for the cell C_i constructed by minimizing the weighted interpolation error [1]

$$\text{err} := \sum_{j\in\mathcal{M}_i}\left[w_{ij}\left(\int_{C_j} P(\vec{x};\phi)\,d\vec{x} - |C_j|\phi_j\right)\right]^2 \tag{4}$$

with respect to the conservativity constraint

$$\int_{C_j} P_i(\vec{x};\phi)\,d\vec{x} = |C_j|\phi_j. \tag{5}$$

The polynomial P_i is computed component by component for $\phi = \rho$, $\phi = \rho u$, $\phi = \rho v$, and $\phi = e$.

The data-dependent weighting is the key point of the WLSQR method The weights w_{ij} are chosen in such a way, that w is big whenever the solution is smooth and w is small when the solution is discontinuous. The single stencil $\mathcal{M}_i$ is selected according to the order of the polynomial P.

[1] Herefrom comes the name of the method - the Weighted Least Square Reconstruction.

2.1. The second order scheme

The formally second order scheme can be obtained by using linear polynomials P_i. For this case, the choice of $\mathcal{M}_i := \mathcal{M}_i^1 = \{j : C_i \cap C_j \neq \emptyset\}$ (i.e. cells touching C_i at least by a vertex) has been tested together with

$$w_{ij} = \sqrt{\frac{h^{-r}}{\left|\frac{\phi_i - \phi_j}{h}\right|^p + h^q}}, \tag{6}$$

with h being the distance between cell centers of C_i and C_j and $p = 4$, $q = -3$, and $r = 3$. The analysis of simplified cases has been carried out in [3] showing the stability of WLSQR interpolation for special discontinuous data.

2.2. The Third Order Scheme

The scheme can be extended to formally third order of accuracy by using quadratic polynomials P_i. It is also necessary to widen the stencil to $\mathcal{M}_i := \mathcal{M}_i^2 = \mathcal{M}_i^1 \cup \{j : C_j \cap \mathcal{M}_i^1 \neq \emptyset\}$ (i.e. the stencil is extended by the cells touching $\mathcal{M}_i^1$). Although there are no analytical results for quadratic reconstruction, the same definition of w has been used successfully.

3. Numerical experiments with the WLSQR scheme

3.1. Ringleb's flow

The piecewise linear and piecewise quadratic reconstruction were numerically analyzed for the case of scalar linear and nonlinear initial value problem in [3]. Next, the more complicated case of the transonic flow in a 2D channel was examined in [3]. However, the order of accuracy was impaired by the polygonal approximation of the boundary. Therefore another numerical experiment is carried out in this article. In order to be able to compare the numerical solution to the exact one, the Ringleb's flow [5] was chosen. The domain was bounded by the axis x, by two streamlines defined by $k = 0.5$ and $k = 0.8$ and by the outlet part defined by $q = 0.4$ (see [5] for the definition of k and q).

The numerical solution was obtained using three levels of unstructured meshes, the coarse one with 1004 triangles, the middle one with 3936 triangles and the fine one with 15780 triangles. Moreover, three different variants WLSQR interpolations were used: **I1B1** with piecewise linear interpolation and boundary approximated by linear segments, **I2B1** with piecewise quadratic interpolation with boundary approximated by linear segments, and **I2B2** with piecewise quadratic interpolation with boundary approximated by second order parabolas. For the I1B1 case the midpoint rule was used for the evaluation of the flux integrals whereas for the I2B1 and I2B2 cases the two-point Gauss integration formula was employed. Figure 1 shows the isolines of Mach number obtained with the I2B2 variant of the scheme on the coarse mesh. Moreover, it compares the distribution of the entropy along the left wall for three variants of the scheme and three meshes.

Figures 2 and 3 show the isomach and isoentropy lines for those 9 cases. Close inspection of the shape of isomach lines shows slight deformation of their shape near the boundary for I1B1 and I2B1 cases. The isolines of the entropy document clearly, that the I2B1 case (i.e. piecewise quadratic interpolation with polygonal boundaries) gives the worst results.

It follows from this test case, that for the scheme with a piecewise quadratic reconstruction the boundary has to be approximated also with higher order of accuracy, otherwise the numerical results can be worse than the results of low-cost second order method.

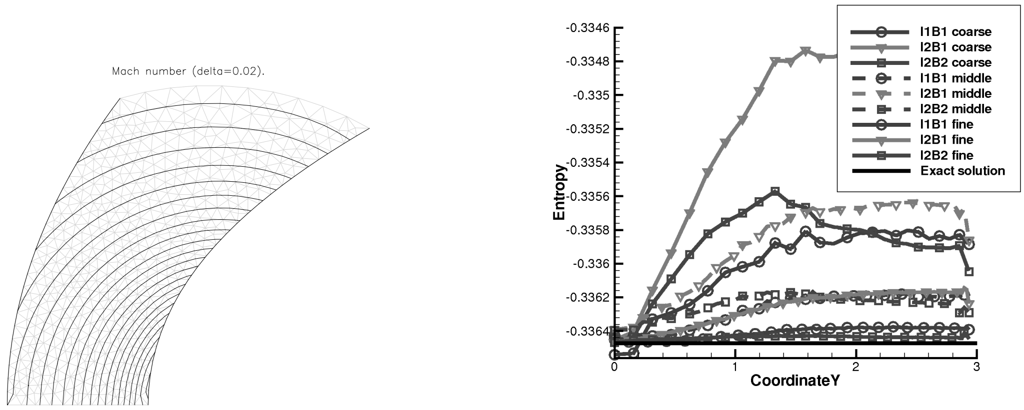


Fig. 1. Isolines of the Mach number and distribution of the entropy along the left wall for Ringleb's flow problem.

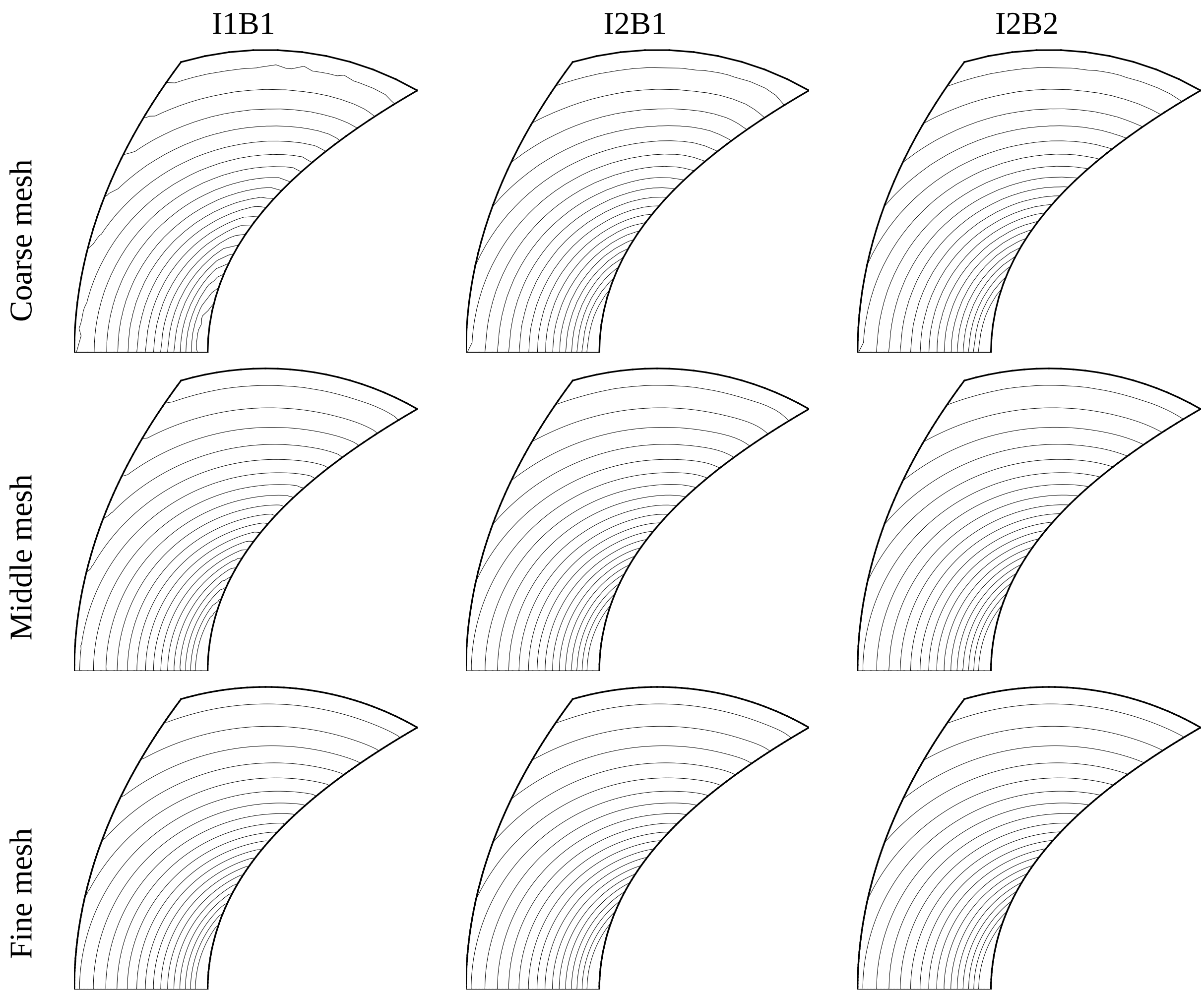


Fig. 2. Isolines of the Mach number for the Ringleb's flow, three different interpolations (columns) and three levels of meshes (rows).

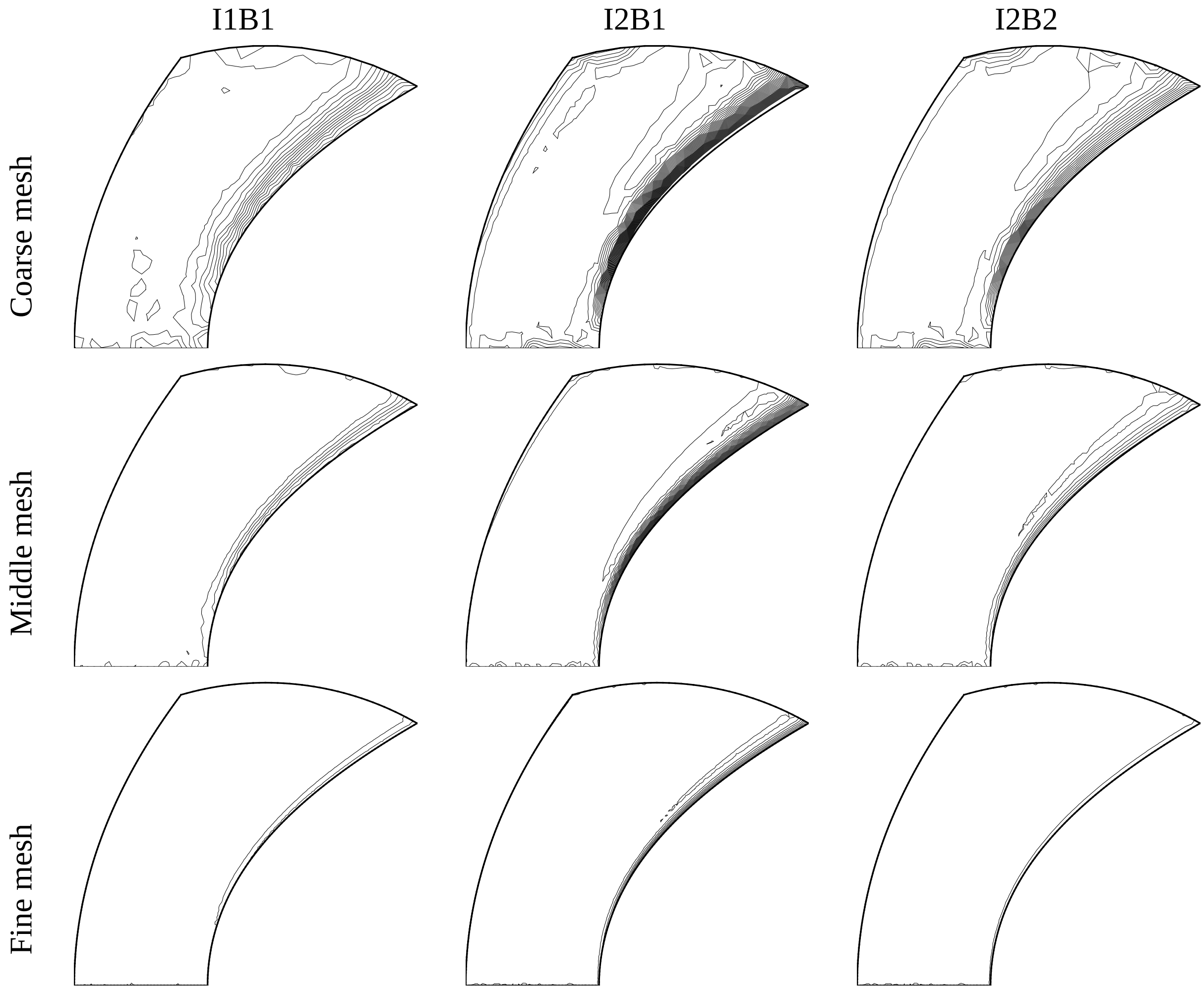


Fig. 3. Isolines of the entropy for the Ringleb's flow, three different interpolations (columns) and three levels of meshes (rows).

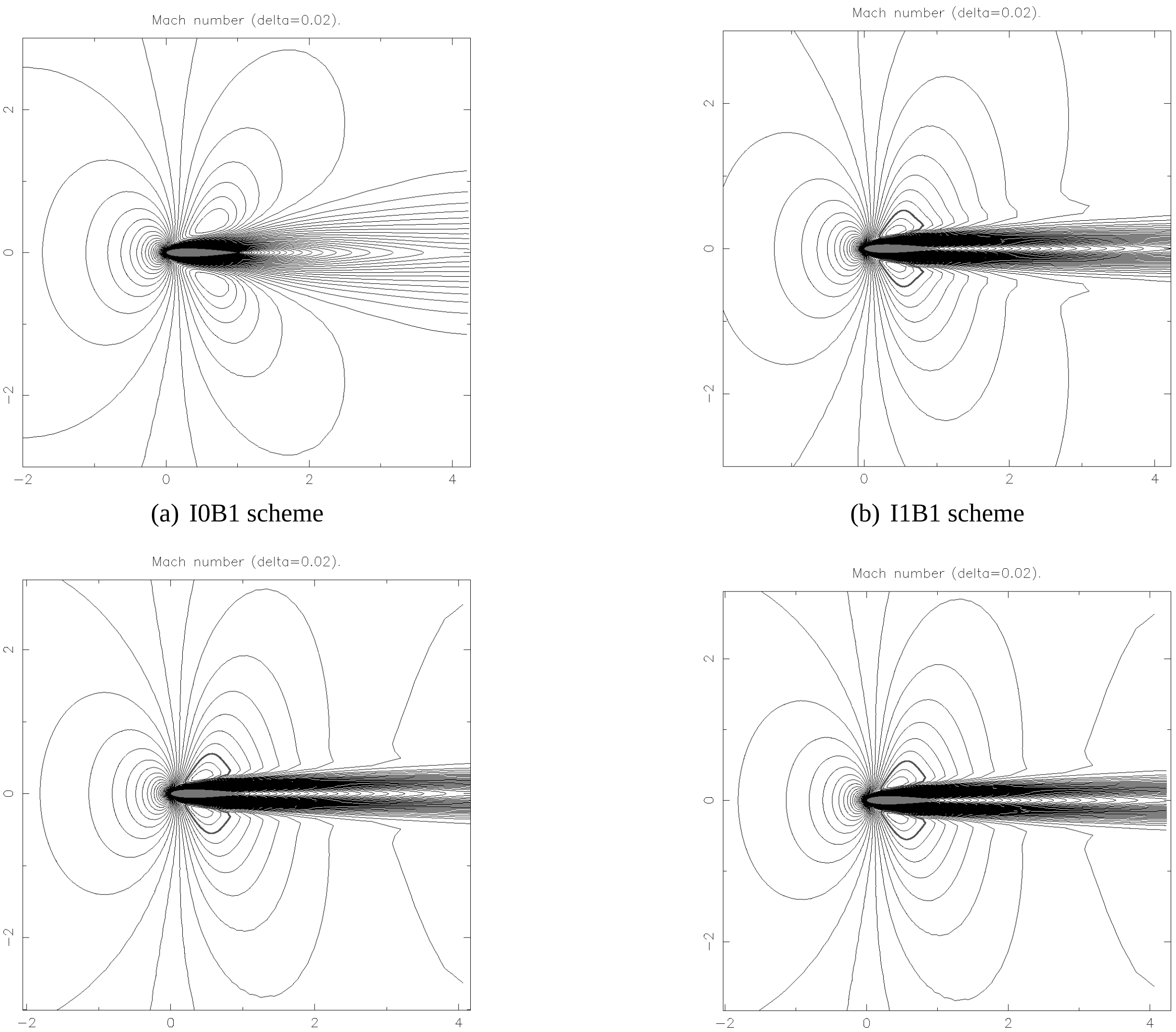
3.2. Laminar flow over the NACA 0012 profile

Another test case is the laminar flow over a profile. The viscous terms in the Navier–Stokes equations are approximated in central fasion using the so called diamond path (or dual cells). This approximation limits the overall order of accuracy to two which impairs the accuracy of I2B1/I2B2 scheme, therefore we call this scheme as mixed order one.

The flow over the NACA 0012 profile is charakterized by the inlet Mach number $M_\infty = 0.8$, inlet angle $\alpha_\infty = 0°$ and Reynolds number $Re = 500$. The results of the basic first order scheme (denoted by I0B1) are compared to the results of the second order scheme (I1B1) and mixed order schemes I2B1 and I2B2 for the same mesh (168×40 cells) at the figure 4. The basic low order scheme clearly suffers from very high amount of numerical viscosity and its results are unacceptable.

3.3. Turbulent flow over the RAE 2822 profile

The transonic flow over the RAE 2822 profile at high Reynolds number is considered here. The turbulence is modeled with the TNT $k-\omega$ model [8]. The inlet Mach number is $M_\infty = 0.73$ and inlet angle is $\alpha_\infty = 2.79°$, and the $Re = 6.5 \cdot 10^6$. The flow is transonic with a shock wave



(a) I0B1 scheme (b) I1B1 scheme

(c) I2B1 scheme (d) I2B2 scheme

Fig. 4. Isolines of the Mach number for the laminar flow over NACA 0012 profile.

at the upper side of the profile. The figure 7 documents clearly the differences between the results obtained with low order and high order schemes. Moreover, it shows, that the second order and mixed order schemes give very similar results. The small difference is visible only at the distribution of friction coefficient c_f.

4. Conclusion

The article presents several results concerning the development of a high order FV scheme. The results of Ringleb's invicid flow problem prove, that the third order scheme can give worse results than the second order one unless a high order approximation of boundaries is used. For the case of viscous flows the importance of high order representation of boundaries is not so important and the mixed order scheme (i.e. third order approximation of convective terms combined with second order approximation of viscous terms) gives very similar results to those obtained with pure second order scheme. Nevertheless, the convergence to steady state can be better for the mixed order scheme.

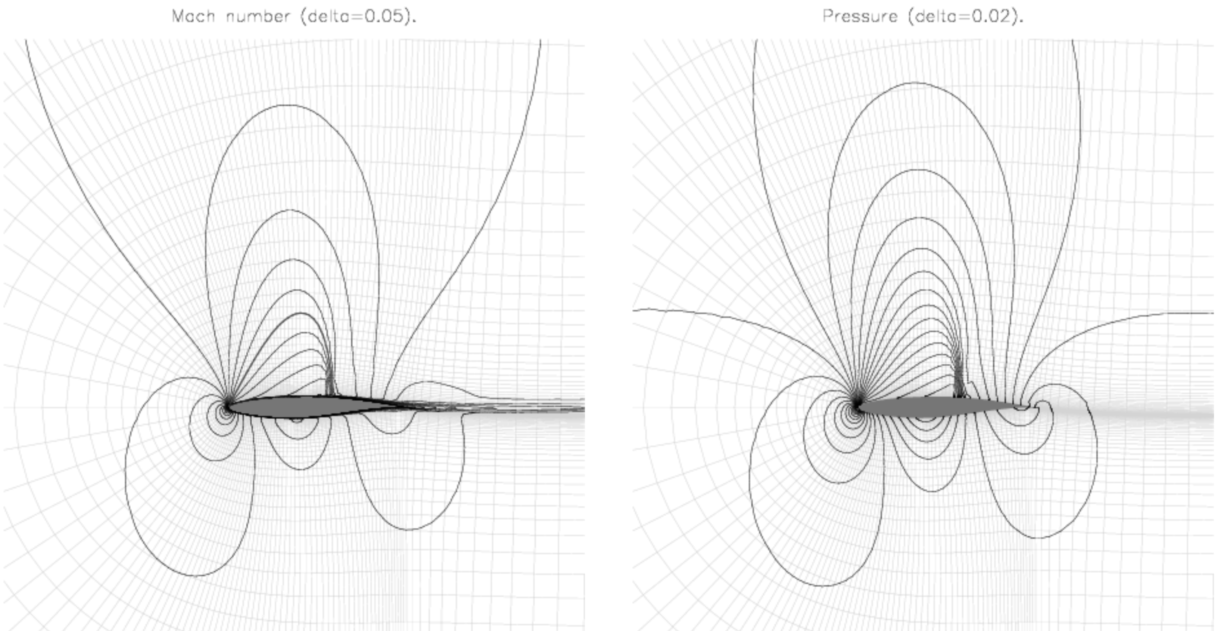


Fig. 5. Isolines of the Mach number and the pressure for the flow around the RAE 2822 profile obtained with the second order scheme (I1B1), $M_\infty = 0.73$, $\alpha_\infty = 2.79°$, $Re = 6.5 \cdot 10^6$.

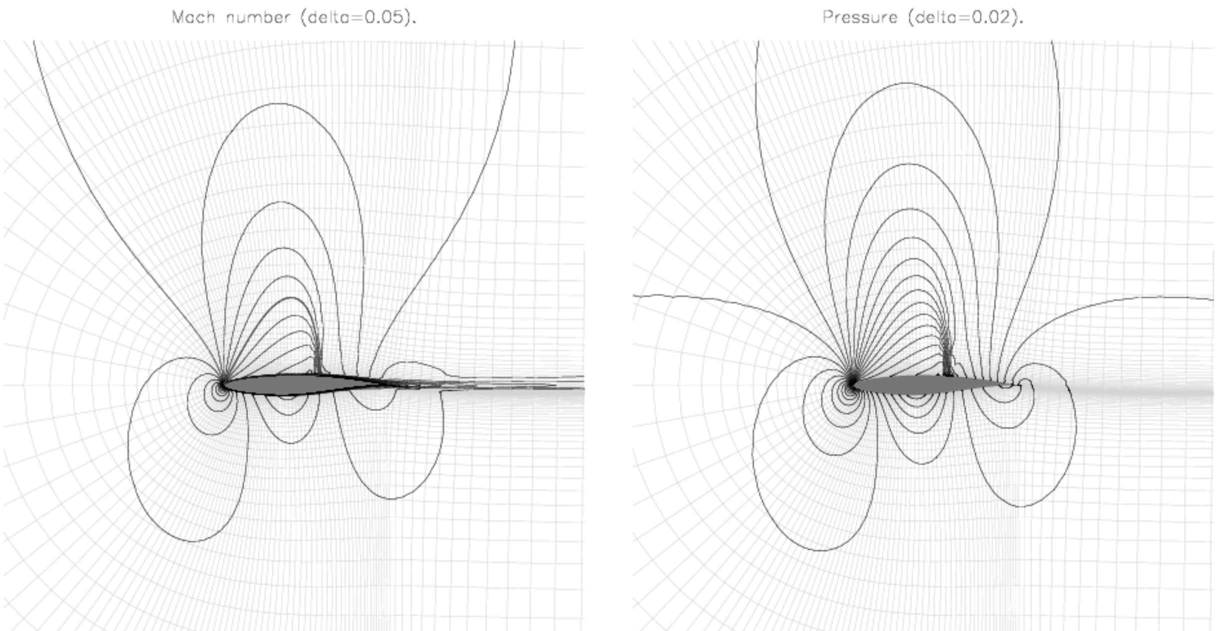


Fig. 6. Isolines of the Mach number and the pressure for the flow around the RAE 2822 profile obtained with the mixed order scheme (I2B1) $M_\infty = 0.73$, $\alpha_\infty = 2.79°$, $Re = 6.5 \cdot 10^6$.

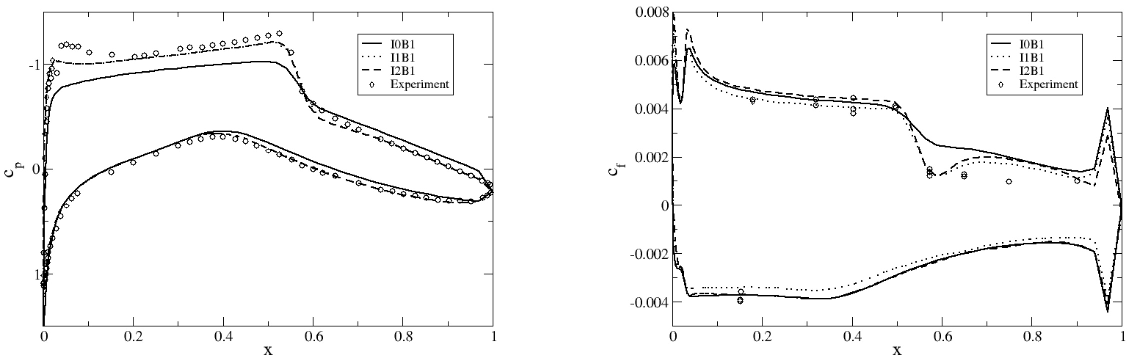


Fig. 7. Distribution of pressure and friction coefficients obtained with the low order (I0B1), the second order (I1B1), and the mixed order (I2B1) schemes.

Acknowledgments

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