

# An improved method for estimating fatigue life under combined stress

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## Abstract

The paper deals with a complicated problem of estimating fatigue lives of machine parts exposed to various kinds of dynamic loading. An origin of the presented method can be found 25 years ago when a simple algorithm of strain energy density cumulation was applied for evaluation of uniaxial harmonic and random fatigue tests. Recently, the method has been adopted for multiaxial loading. Unfortunately, results of its application have shown rather big variance. This paper presents the improved method, which yields much lower scattering of estimated fatigue lives compared with earlier approaches.

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## 1. Introduction

One of general goals of the current grant project, which supports even this contribution, is an improvement of reliability of fatigue life predictions for complex loaded machine parts. Two approaches are pursued in it. One is devoted to improvements of the huge complex computational tool, program PragTic (see [10]), and the other to adjust simple engineering method based on measurements.

The method presented at this conference a year ago gave very promising estimates of fatigue lives of tested specimens for arbitrary loading history [2]. It was based on an uniaxial method, which had been developed many years ago [1], and extended by introducing an artificial effective damaging stress  $\sigma_d$  instead of the original normal stress  $\sigma$ . The effective stress  $\sigma_d$  has been expressed in a complex form by the formula

$$\sigma_d(t) = \sigma(t) + i k_c \tau(t), \quad (1)$$

which is the only one being able to express  $\sigma_d$  as a function of time. Normalized stress energy density  $L_j$  in  $j$ th extreme  $\sigma_{dj}$  has been accepted as measure of partial damage

$$L_j = \frac{\sigma^2(t_j) + k_c^2 \tau^2(t_j)}{R_m^2}, \quad (2)$$

with  $k_c = \sigma_c / \tau_c$ , and the ultimate tensile stress  $R_m$ . The total of  $L_j$  expresses a non-dimensional amount of stress energy density needed for a specimen breakdown caused by a loading aggressivity  $a$ .

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It has been obvious that the amplitude is the key component of the aggressivity of an uniaxial harmonic process. The amplitude is simultaneously the standard deviation of the process extremes. If the same approach is applied to the non-harmonic processes, the aggressivity should somehow depend on distribution of their peaks. Hence, the general non-dimensional form of the aggressivity has been accepted as

$$a = \frac{s_d}{R_m} g(\mu_d), \quad \text{where} \quad \mu_d = \frac{1}{N} \sum_{\forall j} \sigma_{dj}^4 / s_d^4, \quad (3)$$

with  $g(\mu_4)$  as a function of the fourth normalized moment of stress peaks - kurtosis. After processing of Pfeiffer's data published in [11], the non-dimensional aggressivity  $a$  was set in the simplest form for  $g(\mu_4) \doteq \mu_4$ . The first results obtained by processing a limited number of fatigue tests with harmonic and combined loading and presented at [2] have predicted that this has been a good choice. However, recent results (see [3]) have changed our minds. An attempt to process all collected tests of fatigue lives belonging to plain and notched tubes resulted in unsatisfactory deviations of fatigue life estimates from an average fatigue life line

$$L = a^b, \quad (4)$$

in which  $L$  is the total of stress energy densities at extremes  $\sigma_{dj}$ ,  $a$  is the aggressivity of those peaks, and  $b$  expresses a slope of the fatigue life straight line in log-log axes. It has been clear that a modification of the method is necessary should the variance of estimates be diminished.

## 2. Modification of the method

At the beginning, it was believed that the aggressivity  $a$  of loading processes can be expressed in the simple form of equation (3). After many trials with mathematical formula for the aggressivity, it was found that the formula (3) did not express full complexity of the problem. The aggressivity may depend not only on kurtosis of extremes, but also on other statistics of the loading processes. Hence, it has been accepted in the form

$$g(\mu_4) = \mu_4 \left( c_1 + c_2 \frac{s_\sigma}{s_d} + c_3 \frac{s_\tau}{s_d} + c_4 \frac{\sigma_m}{s_d} + c_5 \frac{\tau_m}{s_d} \right), \quad (5)$$

where  $s_\sigma$  and  $s_\tau$  are standard deviations of normal and shear stresses, respectively, and  $\sigma_m$  and  $\tau_m$  are mean stresses. The corrective terms (in brackets) are also non-dimensional. The unknown linear combination coefficients  $c_k$  and the exponent  $b$  of the fatigue life curve can be found by non-linear curve fitting procedure based on minimization of sum of squares  $\mathbf{r}^T \mathbf{r}$ , where residuals derived from eqn (4) are

$$\mathbf{r} = b \log(\mathbf{a}) - \log(\mathbf{L}). \quad (6)$$

Any numerical method for nonlinear least squares can be used for the purpose. The Levenberg-Marquardt-Fletcher algorithm has been applied for.

### 2.1. Levenberg-Marquardt-Fletcher algorithm

Let us have a general overdetermined system of nonlinear algebraic equations

$$\mathbf{f}(\mathbf{x}, \mathbf{c}) \doteq \mathbf{y} \quad \Rightarrow \quad \mathbf{f}(\mathbf{x}, \mathbf{c}) - \mathbf{y} = \mathbf{r}. \quad (7)$$

Here,  $\mathbf{f}(\mathbf{x}, \mathbf{c}) = b \log(\mathbf{a})$ ,  $\mathbf{c} = [b, c_1, c_2, c_3, c_4, c_5]^T$  and  $\mathbf{y} = \log(\mathbf{L})$ , where  $\mathbf{L} = \sum_{\forall j} L_j$ . Its solution, optimal in the least squares sense, is sought by minimizing  $\|\mathbf{r}\|_2^2 = \mathbf{r}^T \mathbf{r}$ .

Necessary conditions for the optimum solution are zero values of partial derivatives of  $\|\mathbf{r}\|_2^2$  due to unknown coefficients  $\mathbf{c}$ , i.e.

$$\frac{\partial \|\mathbf{r}\|_2^2}{\partial \mathbf{c}} = 2 \frac{\partial \mathbf{r}^T}{\partial \mathbf{c}} \mathbf{r} = 2 \mathbf{J}^T \mathbf{r} = 2\mathbf{v} . \quad (8)$$

Elements of  $\mathbf{J}$ , which is called Jacobian matrix, are  $J_{ij} = \frac{\partial r_i}{\partial c_j}$ . Vector  $\mathbf{v}$  should equal zero vector in the point of optimal solution  $\mathbf{c}^*$ . It is sought after  $k$ th iteration in the form

$$\mathbf{c}^{(k+1)} = \mathbf{c}^{(k)} + \Delta \mathbf{c}^{(k)} . \quad (9)$$

Let residuals  $\mathbf{r}(\mathbf{c})$  are smooth functions, then it holds:

$$\mathbf{r}^{(k+1)} = \mathbf{r}^{(k)} + \frac{\partial \mathbf{r}^{(k)}}{\partial \mathbf{c}^{(k)}} \Delta \mathbf{c}^{(k)} + \dots . \quad (10)$$

After some manipulations, the equation for the solution increment takes the form

$$\mathbf{A}^{(k)} \Delta \mathbf{c}^{(k)} - \mathbf{J}^{(k)T} \mathbf{r}^{(k+1)} = -\mathbf{v}^{(k)} , \quad (11)$$

in which  $\mathbf{A} = \mathbf{J}^T \mathbf{J}$ . The solution would be found quite easy should  $\mathbf{r}^{(k+1)}$  be known. Unfortunately, this is not true. It was the reason why Levenberg in [6] substituted the second term in equation (11),  $\mathbf{J}^{(k)T} \mathbf{r}^{(k+1)}$ , by  $\lambda \Delta \mathbf{c}^{(k)}$ . The scalar  $\lambda$  serves for scaling purposes. For  $\lambda = 0$ , the method transforms into fast Newton method, which may diverge, while for  $\lambda \rightarrow \infty$ , the method approaches the stable steepest descent method.

Later, Marquardt [7] changed  $\lambda$  into  $\lambda^{(k)}$ , so that the equation for  $\Delta \mathbf{c}^{(k)}$  became

$$(\mathbf{A}^{(k)} - \lambda^{(k)} \mathbf{I}) \Delta \mathbf{c}^{(k)} = -\mathbf{v}^{(k)} . \quad (12)$$

Values of  $\lambda^{(k+1)}$  are varying in dependence of behavior of iteration process. For slow stable convergence of iterations, the new value  $\lambda^{(k+1)} = \lambda^{(k)}/\nu$  is set, which accelerates the process. Should a sign of divergence be observed, the value is changed into  $\lambda^{(k+1)} = \lambda^{(k)} \nu$ . The usual value for  $\nu$  is 10.

Fletcher improved Marquardt strategy of  $\lambda$  adaptation significantly [5]. He substituted the unity matrix  $\mathbf{I}$  by a diagonal matrix  $\mathbf{D}$  of scales in the formula (12). More to it, he introduced new quotient  $R$  which expresses how forecasted sum of squares agrees with the real one in the current iteration step. If  $R$  falls between preset limits ( $R_{lo}$ ,  $R_{hi}$ ), parameters of iteration do not change, otherwise changes of  $\lambda$  and  $\nu$  follow. Value of  $\lambda$  is halved if  $R > R_{hi}$ . Provided  $\lambda$  becomes lower than a critical value  $\lambda_c$ , it is cleared which causes the next iteration proceeds like in the Newton method. If  $R < R_{lo}$ , parameter  $\nu$  is set so that it holds  $2 \leq \nu \leq 10$ , and if  $\lambda$  were zero, a modification of  $\lambda_c$  and  $\lambda$  follows.

The algorithm has been implemented in MATLAB function *LMFsolve* (see [4]), and applied for processing data from long-term tests of tube specimens.

### 3. Fatigue life testing

#### 3.1. Physical experiments

All tests were performed on tube specimens made of steel ČSN 41 1523.1 in two variants, plain and notched. A sketch of the notched specimen is presented in the fig. 1. The only difference between both types of specimens consists in the fact that the plain tube is without a hole of 3 mm diameter drilled transversally in the middle of the specimen thin part. There were several kinds of tests performed with the specimens:

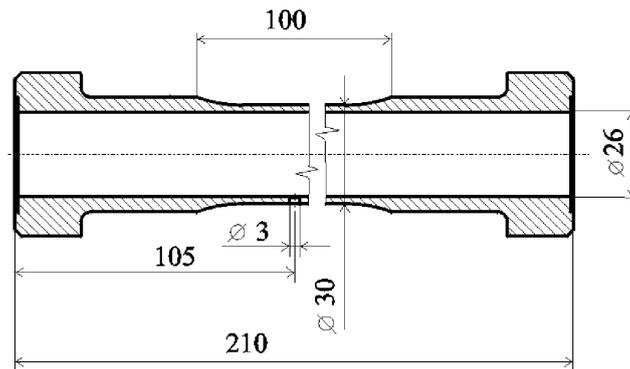


Fig. 1. Notched tube specimen.

- axial harmonic force loading generating harmonic normal stress  $\sigma(t) = \sigma_a \sin(\omega t)$ ,
- harmonic torque loading producing shear stress  $\tau(t) = \tau_a \sin(\omega t)$ ,
- random force or torque loading creating random  $\sigma(t)$  or  $\tau(t)$ , respectively,
- complex loading by a simultaneous combination of random forces and torques.

All loads were generated by computer controlled electro-hydraulic testing system Inova ZUZ 200-1. Parameters of harmonic processes were set manually before the test started. During the test, amplitudes of processes were maintained with high precision by the control system of the machine, and peaks of cycles were counted. Quite different situation raised up for random tests. Even that time series of samples of the non-correlated random loading processes *A* and *B* had been prepared in advance, the real processes were different due to the testing machine and its control system, dynamic properties of which introduce errors which are in the order of 10-20% of peak values. This would completely spoil any estimation of fatigue lives. This was the reason why the time series of one block of both *real* loading processes were measured and stored in files for later processing.

### 3.2. Data structure

It has been clear since the very beginning that the number of tests to be processed will be quite high. This has been a reason why an attention has been paid to formal questions concerning file organization.

A special program named *fatgED* has been built in MATLAB for processing of large groups of mixed kinds of fatigue tests. It is stored in a program directory with all necessary supporting functions. The program directory contains also three subdirectories. All documented experiments on plain and notched tube specimens are recorded in files and stored in a subdirectory *AllData*. Records on processing all files of measured data are stored in subdirectory *Diar*. Diagrams of the processing results are located in subdirectory *Figures*.

The subdirectory *AllData* contains two subdirectories the names of which were created under the following pattern:

$$\langle specimen\ type \rangle \langle material \rangle \langle plain/notch\ sign \rangle .$$

In our case, those two subdirectories have been *T11525h* for plain tubes and *T11523v* for notched tubes. In each of these subdirectories, there are collections of experiment subdirectories containing scaled real time series of applied forces and torques belonging to random loading, and two files. One of the files has a form of function which describes geometry of specimens and some of their material properties. The other file contains text information on parameters of

all harmonic tests. Those parameters are amplitudes and mean values of normal stresses, the same of shears, phase angles between them and number of cycles to damage.

Storing of random experimental data of combined loading is more complicated. A special subdirectory is built for a particular test. Names of the experiment subdirectories are created from the test conditions under the pattern:

$$T\langle force\ process \rangle\langle max.\ force \rangle\langle torque\ process \rangle\langle max.\ torque \rangle\langle plain\text{---}notch\ sign \rangle.$$

For example, the subdirectory name *TA200B140v* says that notched (*v*) tube specimen (*T*) loaded by the process *A* scaled to  $\sigma_{\max} \approx 200$  MPa axially, and by the process *B* scaled to  $\tau_{\max} \approx 140$  MPa in torsion served as a tested object. It should be mentioned that the *real* global extremes are lower due to the above mentioned dynamic errors of the loading. In each of test subdirectories, there are files *F.mat* and *M.mat*, which contain one block each of compressed time series of applied force and torque processes, respectively, and a file *Nexp.m* defining numbers of blocks  $N_{b0}$  spent to initiation of a crack, and  $N_{b1}$  to damage.

## 4. Processing of experiments

### 4.1. Data processing

The initial run of the program *fatgED* is controlled manually. It is enabled by the function *inp* designated for keyboard input supported by prompting default values to input. This feature is very user convenient, because it allows to vary input parameters without editing data rather easily.

Earlier published results of the same authors [3] used formula (3) with  $g(\mu_4) = \mu_4$  for evaluation of aggressivity. The old program has been totally rewritten for improving its structure and exploiting formula (5) for  $g(\mu_4)$ . The program always processes data of one kind of specimens, plain or notched. The user chooses which kind of data will be processed by means of the function *inijob*. His decision also determines which subdirectory from *AllData* will be analyzed. At first, the file with harmonic data is processed very fast. The processing of experiment subdirectories takes much more time, because all complex ranges and damages of one block of loading should be evaluated simultaneously with necessary statistics of the processes.

As soon as the last combined test file is processed, the optimization procedure *LMFsolve* is entered for finding the optimum parameters  $b$  and  $c_k$ ,  $k = 1, 2, \dots, 5$ . At last, a figure with processed results is displayed, and a question issued to the user concerning saving of evaluation results. After positive answer, a diary file as a record of processing is transferred into the subdirectory *Diar*, and a figure with a diagram of all fatigue lives is converted in extended postscript form and stored in the subdirectory *Figures*. Names of these files obey the following pattern

$$\langle specimen\ type \rangle\langle material \rangle\langle plain/notch\ sign \rangle\text{---}\langle version \rangle\text{---}\langle 3\ digits \rangle.\langle ext \rangle,$$

say, *T1523v-A-002.txt*. The first three items are the same as in the subdirectory name. The item  $\langle version \rangle$  determines a program version (*A* in this case),  $\langle 3\ digits \rangle$  is an order of processing the same data, possibly with different parameters, and  $\langle ext \rangle$  is a file extension, *txt* for diary, *eps* for figure.

If a characteristic couple of processes  $\sigma(t)$  and  $\tau(t)$  of the  $T_{\text{test}}$  [s] duration were known, the fatigue life of an average object is quite easily estimated from the formula

$$T = \frac{L}{\sum_{\forall j} L_j} T_{\text{test}}. \quad (13)$$

|                     | plain                 | notched  |
|---------------------|-----------------------|----------|
| $k_c$               | $\sqrt{2.6} = 1.6125$ | 1.5      |
| $b$                 | -12.4065              | -4.498   |
| $c_1$               | 0.5068                | 0.2524   |
| $c_2$               | 0.4018                | 0.0710   |
| $c_3$               | 0.5067                | 0.0955   |
| $c_4$               | 25.3421               | 13.0375  |
| $c_5$               | -66.6030              | -11.4830 |
| $N_{\text{tests}}$  | 26 + 22               | 21 + 25  |
| $\sigma_{\Delta L}$ | 3.04                  | 1.68     |

Tab. 1. Coefficients of the corrective function,  
 $N_{\text{tests}}$  are numbers of plain + notched specimens,  
 $\sigma_{\Delta L}$  is relative standard deviation of  $L$ .

At first, it is necessary to evaluate the aggressivity of the loading with the use of the formula (5), knowing the statistics of stress processes. Then, a total relative energy density  $L$  is obtained from the equation (4) as a point on the best fit line. Processing of a set of  $\sigma(t)$  and  $\tau(t)$  under equation (2) yields partial stress energy densities  $L_j$  in extremes.

It is seen from the fig. 2 that the results are different from those obtained by any uniaxial method for fatigue life estimations based on the Pålmgren [9], Miner [8] and other related methods.

#### 4.2. Discussion of results

The diagrams in fig. 2 contain points of resulting fatigue lives and only few important parameters and values. Each plotted point belongs to one experiment. While circles are displaying results of tests with random combined stress, the other markers belong to uniaxial harmonic loading. The straight line represents the best fit of the set of points in log-log coordinates. Points with attached right arrows correspond to unfinished tests, and have not been included in a set for regression. It is obvious that the variance of plain tube results is almost twice as big as that of notched specimens. The explanation of the phenomenon is rather simple: The main role in fatigue of notched specimens is played by the notch itself. Its stress concentration factor superimposes every other possible effect. This is the reason why a scattering of results is small. On the other hand, effects of material imperfections and surface roughness influence fatigue lives at random which results in higher scattering.

The quantity  $\sigma_{\Delta L}$  from tab. 1 denotes half-bandwidth of expected values of fatigue lives. Should the distribution of fatigue life points be normal in the diagrams of fig. 2, more than 68 % of fatigue lives would lay in a band of a width  $2 \sigma_{\Delta L} L$ , where  $L$  is the average relative energy density corresponding to the loading aggressivity. The save fatigue life for  $100 \alpha$  [%] of objects for  $0 < \alpha < 1$  is given by formula

$$L_\alpha = \frac{L}{u_\alpha \sigma_{\Delta L}}, \tag{14}$$

where  $u_\alpha$  is a quantile of standardized normal distribution. The formula (14) says that e.g. 95% of objects survives the fatigue life  $L_\alpha = L/5$  for plain tubes and  $L_\alpha = L/2.763$  for notched tubes, because  $u_{0.95} = 1.6449$  for  $\alpha = 0.95$ .

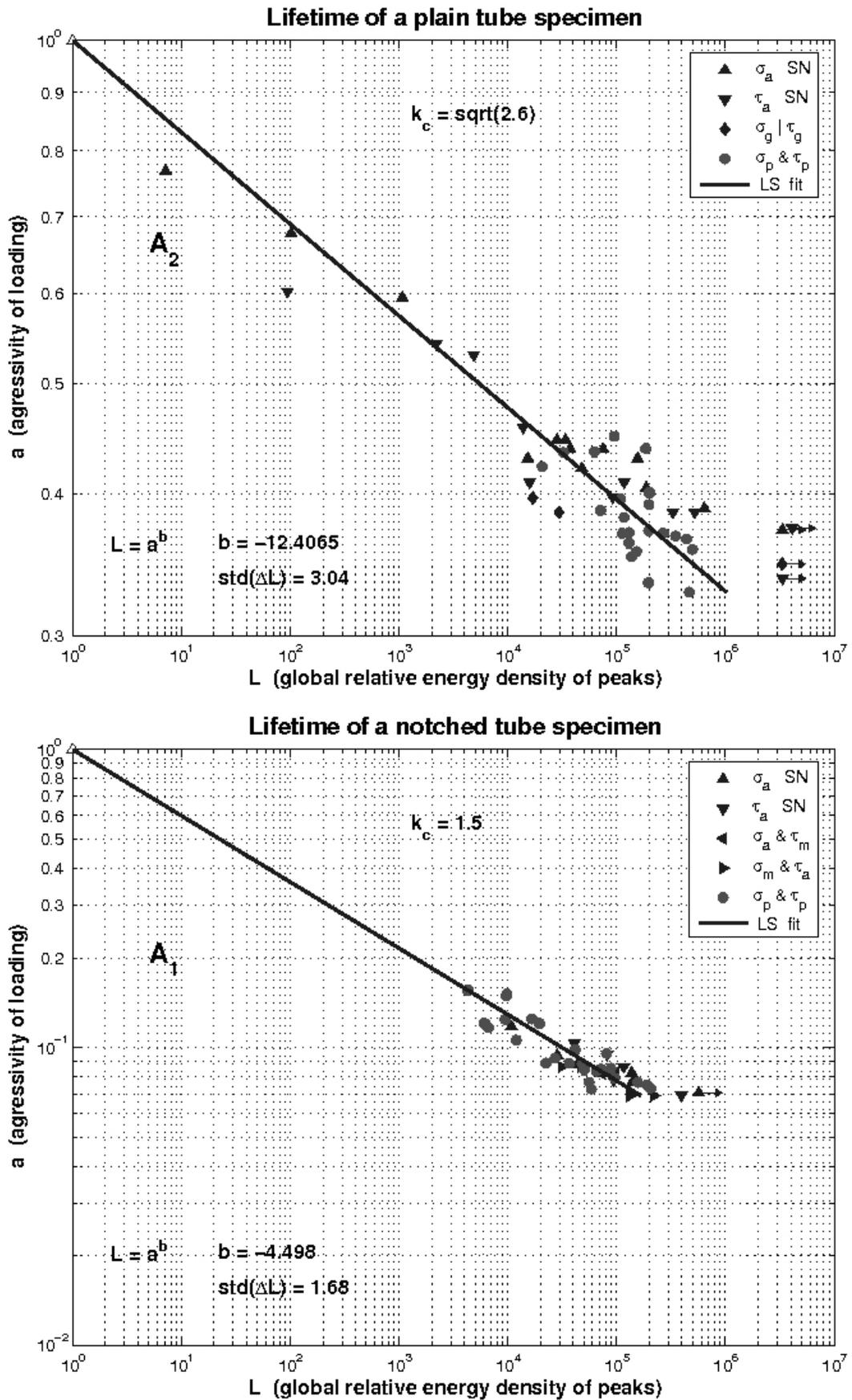


Fig. 2. Fatigue life curves of plain and notched tube specimens.

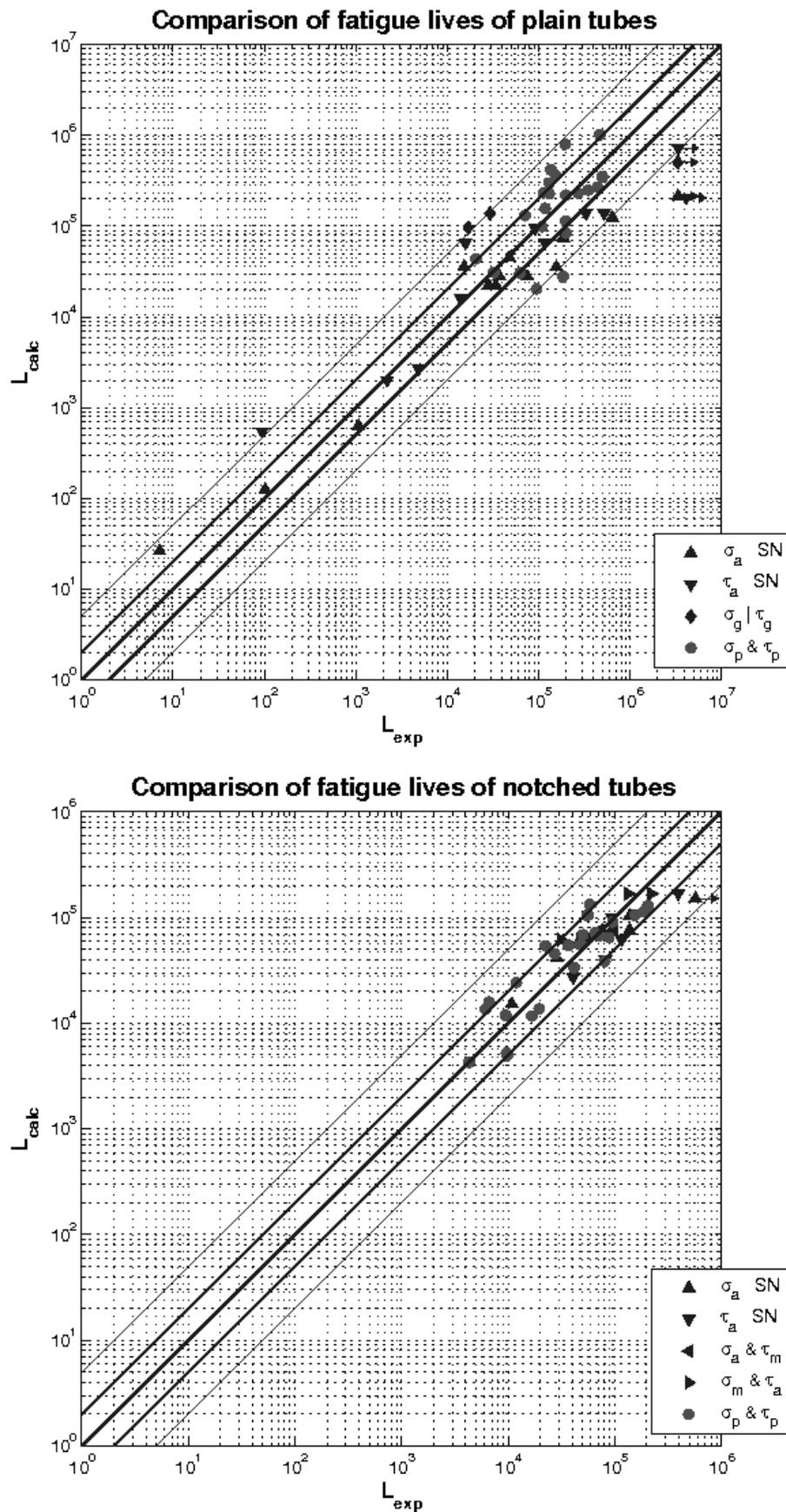


Fig. 3. Calculated vs. experimental fatigue lives with  $\pm \log 2$  and  $\pm \log 5$  bands. Unfinished tests are denoted by right arrows.

As far as markers in all figures are concerned, triangles belong to results obtained by the standard S-N testing of the tube specimens. Results denoted by diamonds and the text  $\sigma_g|\tau_g$  in diagram legends belong to harmonic tests under a special kind of combined loading with combinations  $\sigma_a + \tau_m$  or  $\sigma_m + \tau_a$ .

Fig. 3 contains two diagrams depicting differences between measured and estimated fatigue lives of sets of plain and notched tube specimens. The center line expresses an ideal situation in which measured and estimated fatigue lives are equal. The next pair of lines defines a band in which estimated fatigue life  $L_{\text{calc}} \in (\frac{1}{2}, 2) L_{\text{exp}}$ . The outer straight lines pair borders a band of estimated fatigue lives differing less than five times from the measured values, what means that  $L_{\text{calc}} \in (\frac{1}{5}, 5) L_{\text{exp}}$  in this case. It is obvious that almost all results (but of unfinished tests) have fallen into the  $\mp 5$  band for plain tubes and into the  $\mp 2$  band for notched specimens.

## 5. Conclusions

The paper describes a new approach for the estimation of fatigue lives of objects subject to uniaxial or plane stress of harmonic or random character. The method is based on cumulation of stress energy density in peaks of an effective stress as a complex function of normal and tangential stresses. The cumulated energy density is a function of loading aggressivity. The aggressivity is calculated as a product of the standard deviation of effective stress ranges and their kurtosis in recent works. Both the total energy density and the aggressivity are referred to the material strength limit in order to get non-dimensional quantities. The current paper introduces a multiplicative corrective term into the aggressivity formula.

In spite of much smaller variance yielded by the modified method, one may expect still better results, if more realistic material model was introduced. While the presented method still uses the linear (Hook's) material model, a nonlinear model would be more close to reality. The remaining drawback of the method is, that it does not reflect the directional damaging effects of individual stress ranges yet. An including of this effect does not seem to be very difficult.

The presented method of fatigue test evaluation unifies the process of fatigue life estimation regardless of a kind of loading. It means that the fatigue life of arbitrarily loaded parts might be estimated from the standard S-N curve of a given material. The quality of fatigue life estimation will be even better as soon as further improvements of the method will be implemented.

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