Differential transform method to study free transverse vibration of monoclinic rectangular plates resting on Winkler foundation

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Abstract
This paper analyses free transverse vibration of a monoclinic rectangular plate of uniform thickness resting on Winkler foundation using differential transform method (DTM). Two parallel edges of the plate are taken according to Levy approach i.e., simply supported and other two edges may be either clamped-clamped or clamped-simply supported. This semi-numerical-analytical technique converts the governing differential equation and boundary conditions into a set of algebraic equations. Characteristic equations have been obtained for above two combinations of boundary conditions in the form of infinite series and solved numerically by truncating these equations to finite number of terms. Robustness and convergence of the method is confirmed through numerical results. Two dimensional and three dimensional mode shapes have been plotted for both the cases.

Keywords: differential transform method, monoclinic, rectangular, Winkler foundation

1. Introduction
Recently, solutions of engineering problems have appeared in the literature using differential transform method (DTM). In 1986, Zhou [14] developed it to solve initial value problems occurring in electrical circuits. It works as an alternative approach for getting Taylor series solution. Using this method, the solution of the problem is obtained in the form of a polynomial. So far, the eigenvalue problems have been solved using Frobenius method [3], finite difference method [9], finite element method [11], differential quadrature method [12], Chebyshev collocation technique [7], discrete singular convolution method [10] and Rayleigh-Ritz method [6], etc. DTM seems quite easily applicable for getting the solution of eigenvalue problems. Very few vibration problems have been solved using DTM [1, 2, 5, 8, 13]. In this paper, DTM has been applied to fourth order boundary value problem that represents free transverse vibration of monoclinic thin rectangular plate of uniform thickness resting on Winkler foundation. The first three natural frequencies have been presented for two boundary configurations. This paper has been organized as follows: In section 2, mathematical model of the problem under study is presented. Section 3 presents the respective boundary conditions. Section 4 is concerned with the solution and results of the problem. Conclusions are presented in section 5.

2. Mathematical model of the problem
A monoclinic rectangular plate of uniform thickness h with the domain 0 ≤ x ≤ a, 0 ≤ y ≤ b, where a and b are the length and the breadth of the plate, respectively, is considered. The z-axis
is taken in the perpendicular direction of $xy$-plane. Middle surface of the plate is denoted by $z = 0$. One of the corners of the plate is designated as the origin of the plate. The plate is resting on Winkler foundation having the foundation modulus $K_f$. Two opposite edges $y = 0$ and $y = b$ are taken to be simply supported (see Fig. 1).

Following Kumar and Tomar [7], the differential equation describing the motion of a monoclinic rectangular plate of uniform thickness resting on Winkler foundation is given as follows:

$$
a_0 \frac{d^4 \ddot{w}}{dX^4} + a_1 \frac{d^3 \ddot{w}}{dX^3} + a_2 \frac{d^2 \ddot{w}}{dX^2} + a_3 \frac{d \dot{w}}{dX} + a_4 \ddot{w} = 0, \tag{1}
$$

where

$$
\begin{align*}
a_0 &= 1, \quad a_1 = 0, \quad a_2 = -2\lambda^2(c_{12} + c_{21} + 2c_{66})/c_{11}, \\
a_3 &= 0, \quad a_4 = \lambda^4(c_{22}/c_{11}) + (12K/h^3) - \Omega^2,
\end{align*}
\quad X = \frac{x}{a},
$$

$$
K = \frac{aK_f}{c_{11}}, \quad \lambda^2 = \frac{m^2 \pi^2 a^2}{b^2}, \quad \Omega^2 = \frac{12\rho a^4 \omega^2}{c_{11} h^2}.
$$

Here $c_{11}, c_{12}, c_{21}, c_{22}, c_{66}$ are elastic coefficients, $\rho$ is density of the plate material, $\omega$ is the circular frequency, $K$ is the foundation parameter and $\Omega$ is the frequency parameter.
3. Boundary conditions

Two boundary conditions, namely, C-C and C-S have been considered where first letter represents the boundary condition at the edge \( X = 0 \) and second one at the edge \( X = 1 \). Here, C is used for clamped edge and S for simply supported edge. The edge \( X = 0 \) is clamped and the edge \( X = 1 \) is either clamped or simply supported. The conditions that should be satisfied by clamped and simply supported edges are

\[
\bar{w} = \frac{d\bar{w}}{dX} = 0 \text{ for clamped edge} \tag{2}
\]

and

\[
\bar{w} = \frac{d^2\bar{w}}{dX^2} - \left( \frac{c_{12} + c_{21}}{c_{11}} \right) \lambda^2 \bar{w} = 0 \text{ for simply supported edge.} \tag{3}
\]

4. Solution and results of the problem

The Taylor’s series expansion of a function \( \bar{w}(X) \) may be written as

\[
\bar{w}(X) = \sum_{i=0}^{\infty} (X - X_0)^i \bar{W}_i, \tag{4}
\]

where \( \bar{W}_i = \frac{1}{i!} \left[ \frac{d^i \bar{w}}{dX^i} \right]_{X=X_0} \) is called \( i \)-th order differential transform of \( \bar{w}(X) \) about a point \( X = X_0 \). The series is truncated to finite number of terms i.e., \( N \) while solving practical problems.

Taking the differential transform of equation (1) at \( X_0 = 0 \), we get

\[
a_0 \frac{(i + 4)!}{i!} \bar{W}_{i+4} + a_2 \frac{(i + 2)!}{i!} \bar{W}_{i+2} + a_4 \bar{W}_i = 0, \tag{5}
\]

as differential transform of \( \frac{d^k \bar{w}}{dX^k} \) is given by \( \frac{(i+k)!}{i!} \bar{W}_{i+k} \).

After taking the differential transform, the boundary conditions (2) and (3) may be written as follows:

\[
\sum_{i=0}^{N} (X - X_0)^i \bar{W}_i = 0,
\]

\[
\sum_{i=0}^{N} i(X - X_0)^{i-1} \bar{W}_i = 0 \tag{6}
\]

and

\[
\sum_{i=0}^{N} (X - X_0)^i \bar{W}_i = 0,
\]

\[
\sum_{i=0}^{N} i(i - 1)(X - X_0)^{i-2} \bar{W}_i - \frac{c_{12} + c_{21}}{c_{11}} \lambda^2 \sum_{i=0}^{N} (X - X_0)^i \bar{W}_i = 0. \tag{7}
\]

The equation (5) can be re-written in the following manner

\[
\bar{W}_{i+4} = \frac{R}{(i + 4)(i + 3)} \bar{W}_{i+2} + \frac{S}{(i + 4)(i + 3)(i + 2)(i + 1)} \bar{W}_i, \quad i = 0, 1, 2, 3, \ldots, N, \tag{8}
\]
where
\[ R = \frac{2(c_{12} + c_{21} + 2c_{66})\lambda^2}{c_{11}}, \quad S = \Omega^2 - \left(\frac{c_{22}}{c_{11}}\right)\lambda^4 - (12K/h^3). \]

Now, characteristic equations for both the cases can be obtained by adopting the following mathematical procedure:

Case 1: Clamped at \( X = 0 \) and clamped at \( X = 1 \)

Let \( X_0 = 0 \). At \( X = 0 \), equations (6) become
\[
\begin{align*}
\bar{W}_0 + 0\bar{W}_1 + 0\bar{W}_2 + 0\bar{W}_3 + 0\bar{W}_4 + 0\bar{W}_5 + \ldots & = 0, \\
0\bar{W}_0 + \bar{W}_1 + 0\bar{W}_2 + 0\bar{W}_3 + 0\bar{W}_4 + 0\bar{W}_5 + \ldots & = 0,
\end{align*}
\]
i.e.,
\[ \bar{W}_0 = \bar{W}_1 = 0. \]

Using equation (8), first few terms can be written as follows:
\[
\begin{align*}
\bar{W}_4 &= \frac{R}{12}\bar{W}_2, \\
\bar{W}_6 &= \frac{R}{30}\bar{W}_4 + \frac{S}{360}\bar{W}_2, \\
\bar{W}_8 &= \frac{R}{56}\bar{W}_6 + \frac{S}{1680}\bar{W}_4, \\
\bar{W}_5 &= \frac{R}{20}\bar{W}_3, \\
\bar{W}_7 &= \frac{R}{42}\bar{W}_5 + \frac{S}{840}\bar{W}_3, \\
\bar{W}_9 &= \frac{R}{72}\bar{W}_7 + \frac{S}{3024}\bar{W}_5.
\end{align*}
\]

It is evident from equations (10) that \( \bar{W}_{2i} \) and \( \bar{W}_{2i+1} \) can be represented in terms of \( \bar{W}_2 \) and \( \bar{W}_3 \), respectively.

At \( X = 1 \), equations (6) become
\[
\begin{align*}
\bar{W}_0 + \bar{W}_1 + \bar{W}_2 + \bar{W}_3 + \bar{W}_4 + \bar{W}_5 + \ldots & = 0, \\
0\bar{W}_0 + \bar{W}_1 + 2\bar{W}_2 + 3\bar{W}_3 + 4\bar{W}_4 + 5\bar{W}_5 + \ldots & = 0,
\end{align*}
\]
i.e.,
\[ \bar{W}_0 = \bar{W}_1 = \bar{W}_2 = \bar{W}_3 = \bar{W}_4 = \bar{W}_5 = \ldots = 0. \]

Using (10), equations (11) can be written as follows:
\[
\begin{align*}
p_{11}(\Omega)\bar{W}_2 + p_{12}(\Omega)\bar{W}_3 &= 0, \\
p_{21}(\Omega)\bar{W}_2 + p_{22}(\Omega)\bar{W}_3 &= 0.
\end{align*}
\]

The characteristic equation is obtained from the non-trivial condition of (12) i.e.,
\[
\begin{vmatrix}
p_{11}(\Omega) & p_{12}(\Omega) \\
p_{21}(\Omega) & p_{22}(\Omega)
\end{vmatrix} = 0,
\]
where \( p_{ij}, i, j = 1, 2 \) are polynomials in \( \Omega^2 \).

In particular,
\[
\begin{align*}
p_{11}(\Omega) &= 1 + \frac{R}{12} + \frac{R^2 + S}{360} + \frac{R^3 + 2RS}{20160} + \ldots, \\
p_{12}(\Omega) &= 1 + \frac{R}{20} + \frac{R^2 + S}{840} + \frac{R^3 + 2RS}{60480} + \ldots, \\
p_{21}(\Omega) &= 2 + \frac{4R}{12} + \frac{6(R^2 + S)}{360} + \frac{8(R^3 + 2RS)}{20160} + \ldots, \\
p_{22}(\Omega) &= 3 + \frac{5R}{20} + \frac{7(R^2 + S)}{840} + \frac{9(R^3 + 2RS)}{60480} + \ldots
\end{align*}
\]
Case 2: Clamped at \( X = 0 \) and simply supported at \( X = 1 \)

Differential transform of equations (7) at \( X = 1 \) leads to

\[
\begin{align*}
\ddot{W}_2 + \ddot{W}_3 + \ddot{W}_4 + \ddot{W}_5 + \ldots &= 0, \\
2\ddot{W}_2 + 6\ddot{W}_3 + 12\ddot{W}_4 + 20\ddot{W}_5 + \ldots &= 0.
\end{align*}
\]

Hence, the non-trivial condition of (14) after incorporating (10) provides the following characteristic equation

\[
\begin{vmatrix}
    p_{11}(\Omega) & p_{12}(\Omega) \\
    p_{21}(\Omega) & p_{22}(\Omega)
\end{vmatrix} = 0,
\]

where

\[
\begin{align*}
p_{11}(\Omega) &= 1 + \frac{R}{12} + \frac{R^2 + S}{360} + \frac{R^3 + 2RS}{20160} + \ldots, \\
p_{12}(\Omega) &= 1 + \frac{R}{20} + \frac{R^2 + S}{840} + \frac{R^3 + 2RS}{60480} + \ldots, \\
p_{21}(\Omega) &= 2 + \frac{12R}{12} + \frac{30(R^2 + S)}{360} + \frac{56(R^3 + 2RS)}{20160} + \ldots, \\
p_{22}(\Omega) &= 6 + \frac{20R}{20} + \frac{42(R^2 + S)}{840} + \frac{72(R^3 + 2RS)}{60480} + \ldots
\end{align*}
\]

Displacements of the plates are obtained using the following function:

\[
\ddot{W}(X) = \sum_{i=0}^{N} (X - X_0)^i \ddot{W}_i,
\]

\[
\ddot{W}(X) = X^2\ddot{W}_2 + X^4\ddot{W}_4 + X^6\ddot{W}_6 + X^8\ddot{W}_8 + X^{10}\ddot{W}_{10} + \ldots + X^3\ddot{W}_3 + X^5\ddot{W}_5 + X^7\ddot{W}_7 + X^9\ddot{W}_9 + X^{11}\ddot{W}_{11} + \ldots
\]

\[
= \left[ X^2 + \frac{R}{12} X^4 + \frac{(R^2 + S)}{360} X^6 + \frac{(R^3 + 2RS)}{20160} X^8 + \ldots \right] \ddot{W}_2 + \left[ X^3 + \frac{R}{20} X^5 + \frac{(R^2 + S)}{840} X^7 + \frac{(R^3 + 2RS)}{60480} X^9 + \ldots \right] \ddot{W}_3 - \left[ X^2 + \frac{R}{12} X^4 + \frac{(R^2 + S)}{360} X^6 + \frac{(R^3 + 2RS)}{20160} X^8 + \ldots \right] \ddot{W}_2
\]

\[
\frac{p_{11}(\Omega)}{p_{12}(\Omega)} \left[ X^3 + \frac{R}{20} X^5 + \frac{(R^2 + S)}{840} X^7 + \frac{(R^3 + 2RS)}{60480} X^9 + \ldots \right] \ddot{W}_2
\]

For numerical simulation, rock gypsum has been taken as an example of monoclinic material and the values of elastic constants for the same have been taken from Haussuhl [4] as

\[
c_{11} = 7.859 \times 10^6 \text{ erg/cm}^3, \quad c_{12} = c_{21} = 4.1 \times 10^6 \text{ erg/cm}^3, \quad c_{22} = 6.287 \times 10^6 \text{ erg/cm}^3, \quad c_{66} = 1.044 \times 10^6 \text{ erg/cm}^3.
\]

Apart from it, the values of other parameters considered are

\[
K = 0.01, 0.02, 0.03, 0.04, 0.05, \quad a/b = 0.5, 1.0.
\]

To obtain the values of frequency parameter \( \Omega \), the characteristic equations (13) and (15) have been solved using bisection method with the help of a computer program developed in C++ for different values of aspect ratio \( \lambda = a/b \) and foundation parameter \( K \) for both the boundary configurations. This program was run for different values of \( N \) until we get first
three values of frequency parameter $\Omega$ correct to four decimal places and the value of $N$ has been taken as 36. The value of $m$ has been fixed as 1. The convergence of first three values of frequency parameter $\Omega$ for monoclinic square C-C and C-S plates with increasing number of terms $N$ is shown in Table 1. The desired accuracy of first mode can be achieved by using 26 terms in both the cases and higher modes can be obtained by increasing the number of terms. First three values of frequency parameter $\Omega$ for different combinations of aspect ratio and foundation parameter are presented in Tables 2 and 3 for monoclinic and isotropic plates, respectively. It is concluded that the value of frequency parameter $\Omega$ for C-C plate is greater than that for C-S plate.

Table 1. Convergence of first three values of frequency parameter $\Omega$ for monoclinic square plates for $K = 0.05$

<table>
<thead>
<tr>
<th>$N$</th>
<th>C-C I</th>
<th>C-C II</th>
<th>C-C III</th>
<th>C-S I</th>
<th>C-S II</th>
<th>C-S III</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>21.7242</td>
<td>–</td>
<td>–</td>
<td>19.1580</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>10</td>
<td>29.261</td>
<td>–</td>
<td>–</td>
<td>26.6908</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>15</td>
<td>37.742</td>
<td>290.275</td>
<td>–</td>
<td>33.9770</td>
<td>293.468</td>
<td>–</td>
</tr>
<tr>
<td>20</td>
<td>38.6039</td>
<td>360.043</td>
<td>–</td>
<td>34.7363</td>
<td>360.591</td>
<td>–</td>
</tr>
<tr>
<td>25</td>
<td>38.6257</td>
<td>75.3754</td>
<td>112.7500</td>
<td>34.7544</td>
<td>65.4812</td>
<td>102.8810</td>
</tr>
<tr>
<td>30</td>
<td>38.6257</td>
<td>75.2865</td>
<td>130.1810</td>
<td>34.7544</td>
<td>65.4272</td>
<td>116.4180</td>
</tr>
<tr>
<td>35</td>
<td>38.6257</td>
<td>75.2672</td>
<td>133.5870</td>
<td>34.7544</td>
<td>65.4267</td>
<td>118.2090</td>
</tr>
<tr>
<td>36</td>
<td>38.6257</td>
<td>75.2672</td>
<td>133.5870</td>
<td>34.7544</td>
<td>65.4267</td>
<td>118.2090</td>
</tr>
</tbody>
</table>

Table 2. First three values of frequency parameter $\Omega$ for monoclinic plates

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>C-C</th>
<th>C-S</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>mode</td>
<td>0.5</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>-----</td>
</tr>
<tr>
<td>0.00</td>
<td>I</td>
<td>24.1786</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>64.0740</td>
</tr>
<tr>
<td></td>
<td>III</td>
<td>123.5360</td>
</tr>
<tr>
<td>0.01</td>
<td>I</td>
<td>26.5444</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>65.0037</td>
</tr>
<tr>
<td></td>
<td>III</td>
<td>124.0210</td>
</tr>
<tr>
<td>0.02</td>
<td>I</td>
<td>28.7159</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>65.9203</td>
</tr>
<tr>
<td></td>
<td>III</td>
<td>124.5040</td>
</tr>
<tr>
<td>0.03</td>
<td>I</td>
<td>30.7344</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>66.8243</td>
</tr>
<tr>
<td></td>
<td>III</td>
<td>124.9850</td>
</tr>
<tr>
<td>0.04</td>
<td>I</td>
<td>32.6283</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>67.7162</td>
</tr>
<tr>
<td></td>
<td>III</td>
<td>125.4640</td>
</tr>
<tr>
<td>0.05</td>
<td>I</td>
<td>34.4181</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>68.5965</td>
</tr>
<tr>
<td></td>
<td>III</td>
<td>125.9410</td>
</tr>
</tbody>
</table>
Table 3. First three values of frequency parameter $\Omega$ for isotropic ($c_{12} + c_{21} + \frac{2c_{66}}{c_{11}} = v$, $\frac{c_{22}}{c_{11}} = 1$, $c_{11} = \frac{E}{1-\nu}$) plates

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>C-C</th>
<th>C-S</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>23.815 6</td>
<td>28.950 9</td>
</tr>
<tr>
<td>II</td>
<td>63.534 5</td>
<td>69.327 0</td>
</tr>
<tr>
<td>III</td>
<td>122.929 0</td>
<td>129.090 0</td>
</tr>
<tr>
<td>0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>26.214 2</td>
<td>30.954 0</td>
</tr>
<tr>
<td>II</td>
<td>64.472 0</td>
<td>70.187 2</td>
</tr>
<tr>
<td>III</td>
<td>123.417 0</td>
<td>129.554 0</td>
</tr>
<tr>
<td>0.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>28.411 0</td>
<td>32.835 2</td>
</tr>
<tr>
<td>II</td>
<td>65.396 0</td>
<td>71.036 9</td>
</tr>
<tr>
<td>III</td>
<td>123.902 0</td>
<td>130.017 0</td>
</tr>
<tr>
<td>0.03</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>30.449 7</td>
<td>34.614 3</td>
</tr>
<tr>
<td>II</td>
<td>66.307 1</td>
<td>71.876 5</td>
</tr>
<tr>
<td>III</td>
<td>124.385 0</td>
<td>130.477 0</td>
</tr>
<tr>
<td>0.04</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>32.360 2</td>
<td>36.306 4</td>
</tr>
<tr>
<td>II</td>
<td>67.205 9</td>
<td>72.706 5</td>
</tr>
<tr>
<td>III</td>
<td>124.866 0</td>
<td>130.936 0</td>
</tr>
<tr>
<td>0.05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>34.164 1</td>
<td>37.923 0</td>
</tr>
<tr>
<td>II</td>
<td>68.092 8</td>
<td>73.527 1</td>
</tr>
<tr>
<td>III</td>
<td>125.346 0</td>
<td>131.394 0</td>
</tr>
</tbody>
</table>

Also, frequencies of monoclinic plates are greater than those for isotropic plates for same values of parameters. Further, it increases with the increasing values of foundation parameter $K$ and aspect ratio $a/b$. More importantly, the difference in the values of frequency parameter $\Omega$ for first mode of vibration for monoclinic C-C ($a/b = 0.5$, $K = 0.0, 0.01, 0.02, 0.03, 0.04, 0.05$) and C-S ($a/b = 1.0$, $K = 0.0, 0.01, 0.02, 0.03, 0.04, 0.05$) plates is not considerable. Same conclusion is true for isotropic plate. The percentage variations in the value of frequency parameter are more for C-S plates than those for C-C plates and these percentage variations decrease with the increasing value of $K$ when material changes from isotropic to monoclinic.

The percentage variations in the value of frequency parameter are 1.5, 1.3, 1.1, 0.9, 0.8, 0.7 when $K$ changes from 0.0 to 0.05 for first mode of vibration ($a/b = 0.5$ and C-C plate). These percentage variations are 0.8 and 0.5 for second and third modes, respectively, for all $K$. These variations increase with increasing value of $a/b$. Displacements have been calculated using equation (17).

Two dimensional and three dimensional mode shapes of C-C and C-S plates for $K = 0.01$, $a/b = 1$ have been depicted in Figs. 2–4. Three dimensional mode shapes have been plotted using MATLAB software.
Fig. 2. Normalized displacements of C-C monoclinic plate for $K = 0.01, \ a/b = 1$. First mode (□), second mode (△) and third mode (×)

Fig. 3. Normalized displacements of C-S monoclinic plate for $K = 0.01, \ a/b = 1$. First mode (□), second mode (△) and third mode (×)
Fig. 4. First three mode shapes of (i) C-C and (ii) C-S monoclinic plates for $K = 0.01$, $a/b = 1$ using $w(x, y) = \bar{w}(x/a) \sin(m\pi y/b)$

5. Conclusions

Differential transform method is successfully applied to analyze free transverse vibration of monoclinic rectangular plates of uniform thickness resting on Winkler foundation. The two opposite edges of the plate are assumed to be simply supported. Two boundary conditions namely, clamped and simply supported have been taken on one of the other two parallel edges, keeping the other edge clamped. Characteristic equations have been obtained in the form of infinite series. The series have been truncated to finite number of terms and solved numerically to obtain first three natural frequencies using a computer program developed by the author in
C++ language. Displacements have been calculated and demonstrated in two dimensions as well as three dimensions. Analysis shows that present method performed really well for monoclinic plates in terms of simplicity and efficiency.

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References


