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Thermoelastic wave propagation in laminated composites plates

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Abstract

The dispersion of thermoelastic waves propagation in an arbitrary direction in laminated composites plates is studied in the framework of generalized thermoelasticity in this article. Three dimensional field equations of thermoelasticity with relaxation times are considered. Characteristic equation is obtained on employing the continuity of displacements, temperature, stresses and thermal gradient at the layers' interfaces. Some important particular cases such as of free waves on reducing plates to single layer and the surface waves when thickness tends to infinity are also discussed. Uncoupled and coupled thermoelasticity are the particular cases of the obtained results. Numerical results are also obtained and represented graphically.

Keywords: dispersion, thermoelasticity, laminated, thermal relaxation times, composites plates

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1. Introduction

Increasing use of advanced composites as important structural components in modem high speed aircraft, missile, marine vehicles, and other aerospace structures, and various other applications has led to widespread research activities in the field of composite materials. Composites consist of different materials, so they are inhomogeneous and anisotropic. Different mechanical and thermal properties between constituents of such composites structures, like temperature changes, can generate residual stresses, which may lead to interface de-bonding. A possible failure of the system has intensified the need to study the thermoelastic wave propagation, especially in the form of precise numerical calculations. Consequently, it is of interest to investigate the feasibility of nondestructively, monitoring thermal, mechanical and aging in composites.

Extensive review on the dynamic behavior of anisotropic plate theories can be found in [1] and [14] and problems of wave propagation in periodically layered anisotropic media have been considered and studied in [16,28] and [3]. Dynamic behavior of the problems on the theories of laminated and composite plates have been investigated by authors [12] and [18–23]. Reasonable number of investigations of such advanced materials and their analysis also have been reported in [10,19]. In [15] a transfer matrix technique to obtain the dispersion relation curves of elastic waves propagating in multilayered anisotropic media i.e., composite laminate is developed and detailed review on the wave propagation in layered anisotropic media is studied in [11]. In [9], general problem of thermoelastic waves in anisotropic periodically laminated composites in thermoelasticity is studied.

Theory of thermoelasticity is well established, one can see the works in references [17] and [5]. Literature in this field is rather large to account for the phenomena involving the finite propagation velocity of the thermal wave, and can confer with the reference [4]. These modified

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coupled theories of thermoelasticity are based on hyperbolic-type equations for temperature and are closely connected with the theories of *second sound*, which consider heat propagation as a wavelike phenomenon. In the literature, addressing *linear* theories with relaxation time, most attention is given to the models formulated in [13] and [8]. Theory in [13] called Lord and Shulman (LS) theory is based on a modified Fourier's Law of heat conduction with one relaxation time to dictate the relaxation of thermal propagation, as well as the rate of change of strain rate and the rate of change of heat generation. Green and Lindsay (GL) theory is based on a rigorous treatment of thermodynamics, and a form of the entropy inequality. The literature dedicated to hyperbolic thermoelastic models is quite large and its detailed review can be found in [6, 7].

Theory of generalized thermoelasticty [13] is extended to anisotropic heat conducting elastic materials by [2], and hence it is valid for both isotropic and anisotropic bodies. The propagation of harmonic waves in a laminated composite plate consisting of an arbitrary number of layeres is studied in [9]. Various problems of infinite plates in the context of generalized theories thermoelasticity and the propagation of waves in layered anisotropic media in generalized thermoelasticity is investigated [24–27]. Yamada and Nasser [29] have studied harmonic wave's propagation direction in orthotropic composites.

In this article propagation of thermoelastic waves in layered laminated composites, where the direction of the corresponding harmonic waves makes an arbitrary angle with respect to the layers is examined in the context of generalized thermoelasticity with two thermal relaxation times. Three dimensional field equations of thermoelasticity are considered for this study and the corresponding characteristic equation is obtained on employing the continuity of displacements, temperature, thermal stresses and thermal gradient at the layers' interface. Some important particular cases such as of free waves on reducing plates to single layer and the surface waves when thickness tends to infinity are also discussed. Numerical results are also obtained and represented graphically.

2. Formulation

Consider a set of Cartesian coordinate system $x_i = (x_1, x_2, x_3)$ in such a manner that x_3-axis is normal to the layering. The basic field equations of generalized thermoelasticity for an infinite generally anisotropic thermoelastic medium at uniform temperature T_0 in the absence of body forces and heat sources are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{1}$$

$$K_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} - \rho C_e \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} \right) = T_0 \beta_{ij} \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial t} \right). \tag{2}$$

Constitutive relations for anisotropic materials in the context of generalized thermoelasticity are following:

$$\sigma_{ij} = c_{ijkl} e_{kl} - \beta_{ij}(T + \tau_1 \dot{T}), \tag{3}$$

$$\beta_{ij} = c_{ijkl} \alpha_{kl}, \quad i, j, k, l = 1, 2, 3, \tag{4}$$

where ρ is the density of the nth layer, t is time, u_i is the displacement in the x_i direction, K_{ij} are the thermal conductivities, σ_{ij} and e_{ij} are the stress and strain tensor respectively, C_e is the specific heat at constant strain, β_{ij} are thermal moduli, α_{ij} is the thermal expansion tensor, T

is temperature, and c_{ijkl} is the fourth order tensor of the elasticity. The quantities c_{ijkl}, α_{ij}, β_{ij} satisfy the symmetry conditions

$$c_{ijkl} = c_{klij} = c_{ijlk} = c_{jikl}, \qquad \alpha_{ij} = \alpha_{ji}, \qquad \beta_{ij} = \beta_{ji}. \tag{5}$$

The parameter τ_1 and τ_0 are the thermal-mechanical relaxation time and the thermal relaxation time of the GL theory and satisfy the inequality$\tau_1 \geq \tau_0 \geq 0$. Strain-displacement relation is

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{6}$$

In addition, at the interface between two layers the tractions, temperature gradient, displacements and temperature must be continuous.

3. Analysis

For harmonic waves propagating in an arbitrary direction, the displacements components u_1, u_2, u_3 and temperature T are written as

$$(u_1, u_2, u_3, T) = \{U_1(x_3), U_2(x_3), U_3(x_3), U_4(x_3)\}\mathrm{e}^{\mathrm{i}\xi(l_1x_1+l_2x_2+l_3x_3-ct)}, \tag{7}$$

where ξ is the wave number, $c = \omega/\xi$ is the phase velocity, $i = \sqrt{-1}$, ω is the circular frequency, l_1, l_2 and l_3 are the direction cosine defining the propagation direction as in Fig. 1.

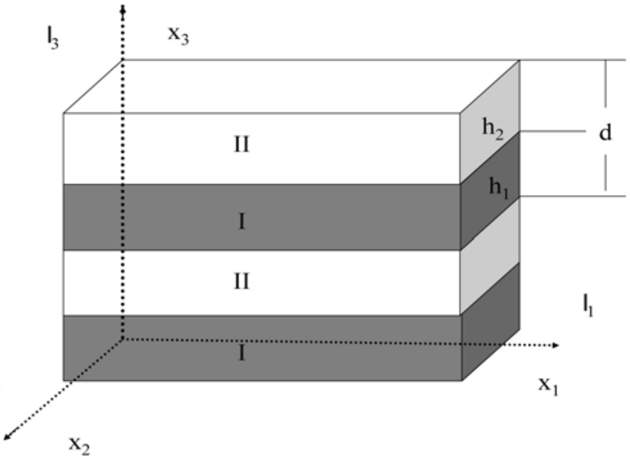

Fig. 1. Two-phase orthotropic layered thermoelastic composite plate. The direction of the propagation vector are denoted as l_1, l_2 and l_3

U_j and T are the constants related to the amplitudes of displacement and temperature, Floquet's theory requires functions U_j ($j = 1, 2, 3$ and 4) to have the same periodicity as the layering. Hence the problem is reduced to that of one pair of layers, where

$$U_j = \bar{U}_j\mathrm{e}^{-\mathrm{i}\xi(l_3+\alpha)x_3}, \quad j = 1, 2, 3, 4, \tag{8}$$

where $\bar{U}_j$ are constants. On substitution of Eq. (8) into Eqs. (1)–(2), via (3)–(6) and specializing the equations for orthotropic media, it follows that

$$M_{mn}(\alpha)\bar{U}_n = 0, \quad m, n = 1, 2, 3, 4, \tag{9}$$

where

$$\begin{aligned}
M_{11} &= (l_1^2 + l_2^2\bar{c}_{66} + \alpha^2\bar{c}_{55} - \zeta^2), \quad M_{12} = (\bar{c}_{12} + \bar{c}_{66})l_1 l_2, && (10)\\
M_{13} &= -(\bar{c}_{13} + \bar{c}_{55})l_1\alpha, \quad M_{14} = l_1, &&\\
M_{22} &= (l_1^2\bar{c}_{66} + l_2^2\bar{c}_{22} + \alpha^2\bar{c}_{44} - \zeta^2), \quad M_{23} = -(\bar{c}_{23} + \bar{c}_{44})l_2\alpha, \quad M_{24} = \bar{\beta}_2 l_2, &&\\
M_{33} &= (l_1^2\bar{c}_{55} + l_2^2\bar{c}_{44} + \alpha^2\bar{c}_{33} - \zeta^2), \quad M_{34} = -\bar{\beta}_3\alpha, \quad M_{41} = \varepsilon\omega^*\zeta^2 l_1\tau_g, &&\\
M_{42} &= \varepsilon\omega^*\zeta^2 l_2\bar{\beta}_2\tau_g, \quad M_{43} = -\varepsilon\omega^*\zeta^2\alpha\bar{\beta}_3\tau_g, \quad M_{44} = l_1^2 + \bar{K}_2 l_2^2 + \bar{K}_3\alpha^2 - \omega^*\zeta^2\tau, && (11)
\end{aligned}$$

where $\zeta^2 = \frac{\rho c^2}{c_{11}}$, $\omega^* = \frac{c_{11}C_e}{K_1}$, $\varepsilon = \frac{\beta_1^2 T_0}{\rho C_e c_{11}}$, and $\tau_g = \tau_1 + i/\omega$, $\tau = \tau_0 + i\omega$. The existence of nontrivial solutions for $\bar{U}_j$ $(j = 1, 2, 3, 4)$ demands the vanishing of the determinant in Eqs. (9), and yields the eighth degree polynomial equation

$$\alpha^8 + A_1\alpha^6 + A_2\alpha^4 + A_3\alpha^2 + A_4 = 0, \tag{12}$$

where the coefficients A_1, A_2, A_3 and A_4 are

$$\begin{aligned}
A_1 &= [Q_1\omega^*\varepsilon\tau_g\zeta^2 + P_1\bar{K}_3 + c_{33}c_{44}c_{55}(l_1^2 + l_2^2\bar{K}_2 - \omega^*\tau\zeta^2)]/\Delta,\\
A_2 &= [Q_2\omega^*\varepsilon\tau_g\zeta^2 + P_2\bar{K}_3 + P_1(l_1^2 + \bar{K}_2 l_2^2 - \omega^*\tau\zeta^2)]/\Delta,\\
A_3 &= [Q_3\omega^*\varepsilon\tau_g\zeta^2 + P_3\bar{K}_3 + P_2(l_1^2 + l_2^2\bar{K}_2 - \omega^*\tau\zeta^2)]/\Delta,\\
A_4 &= [Q_4\omega^*\varepsilon\tau_g\zeta^2 + P_3(l_1^2 + l_2^2\bar{K}_2 - \omega^*\tau\zeta^2)]/\Delta,\\
P_1 &= [(c_{22}c_{33} - 2c_{23}c_{44} - c_{23}^2)c_{55} + c_{33}c_{44}c_{66}]l_2^2 + [(c_{33} - 2c_{13}c_{55} - c_{13}^2)c_{44} + c_{33}c_{55}c_{66}]l_1^2 -\\
&\quad (c_{33}c_{44} + c_{33}c_{55} + c_{44}c_{55})\zeta^2,\\
P_2 &= [(c_{33} - 2c_{13}c_{55} - c_{13}^2)c_{66} + c_{44}c_{55}]l_1^4 + [(c_{22}c_{33} - 2c_{23}c_{44} - c_{23}^2)c_{66} + c_{22}c_{55}c_{44}]l_2^4 +\\
&\quad [-c_{12}^2c_{33} - 2(c_{33}c_{44} - c_{66}c_{23}c_{55} - c_{12}c_{44}c_{55} + c_{13}c_{22}c_{55} - 2c_{44}c_{55}c_{66} - c_{13}c_{44}c_{66} +\\
&\quad c_{12}c_{33}c_{66} - c_{12}c_{13}c_{44} - c_{13}c_{23}c_{66} - c_{12}c_{23}c_{55} - c_{12}c_{13}c_{23}) - c_{13}^2c_{22} + c_{22}c_{33} - c_{23}^2]l_1^2l_2^2 +\\
&\quad [(2c_{13}c_{55} - c_{66}c_{33} - c_{55}c_{44} - c_{44} - c_{33} - c_{66}c_{55} + c_{13}^2)l_1^2 +\\
&\quad (2c_{23}c_{44} + c_{23}^2 - c_{22}c_{33} - c_{22}c_{55} - c_{66}c_{44} - c_{55}c_{44} - c_{33}c_{66})l_2^2 + (c_{33} + c_{44} + c_{55})\zeta^4]\zeta^2,\\
P_3 &= (c_{55}l_1^2 + c_{44}l_2^2 - \zeta^2)\{[(1 + c_{66})l_1^2 + (c_{22} + c_{66})l_2^2]\zeta^2 - \zeta^4 +\\
&\quad [(2c_{22}c_{66} + c_{12}^2 - c_{22})c_{55}]l_1^2l_2^2 - c_{22}c_{66}l_2^4 - c_{66}l_1^4\},\\
&\quad \Delta = c_{33}c_{44}c_{55}\bar{K}_3, \quad Q_1 = -c_{44}c_{55}\bar{\beta}_3^2,\\
Q_2 &= (c_{55} + c_{44})\bar{\beta}_3^2\zeta^2 + [2(c_{13}c_{44} + c_{44}c_{55})\bar{\beta}_3 - c_{33}c_{44} - (c_{44} + c_{66}c_{55})\bar{\beta}_3^2]l_1^2 +\\
&\quad [2(c_{23}c_{55} + c_{44}c_{55})\bar{\beta}_2\bar{\beta}_3 - c_{33}c_{55}\bar{\beta}_2^2 - (c_{22}c_{55} + c_{44}c_{66})\bar{\beta}_3^2]l_2^2,\\
Q_3 &= ((1 + c_{66})\bar{\beta}_3^2 + (c_{44} + c_{33}) - 2(c_{55} + c_{13})\bar{\beta}_3)l_1^2\zeta^2 +\\
&\quad ((c_{22} + c_{66})\bar{\beta}_3^2 + 2(c_{33} + c_{55})\bar{\beta}_2^2 - 2(c_{23} + c_{44})\bar{\beta}_2\bar{\beta}_3)l_2^2\zeta^2 +\\
&\quad [(-\bar{\beta}_3^2 + 2(c_{55} + c_{13})\bar{\beta}_3 - c_{33})c_{66} - c_{44}c_{55}]l_1^4 +\\
&\quad [-c_{22}c_{66}\bar{\beta}_3^2 - (c_{33}c_{66} + c_{44}c_{55})\bar{\beta}_2^2 + 2(c_{44}c_{66} + c_{66}c_{23})\bar{\beta}_2\bar{\beta}_3]l_2^4 +\\
&\quad [(c_{23}^2 + 2c_{23}c_{44} - c_{22}c_{33}) + 2(c_{22}c_{55} + c_{13}c_{22} - c_{12}c_{44} - c_{44}c_{66} - c_{12}c_{23} - c_{23}c_{66})\bar{\beta}_3 +\\
&\quad 2(c_{12}c_{13} + c_{33}c_{66} - c_{44}c_{55} - c_{13}c_{23} - c_{23}c_{55} - c_{13}c_{44})\bar{\beta}_2 +\\
&\quad (c_{13}^2 - c_{33} + 2c_{13}c_{55})\bar{\beta}_2^2 + (c_{12}^2 - c_{22} + 2c_{12}c_{66})\bar{\beta}_3^2 +\\
&\quad 2(c_{23} - c_{13}c_{66} + c_{44} - c_{12}c_{55} - c_{66}c_{55} - c_{12}c_{13})\bar{\beta}_3\bar{\beta}_2]l_1^2l_2^2 - \bar{\beta}_3\zeta^4,\\
Q_4 &= (c_{55}l_1^2 + c_{44}l_2^2 - \zeta^2)[(l_1^2 + \bar{\beta}_2^2 l_2^2)\zeta^2 + (-\bar{\beta}_2^2 + 2\bar{\beta}_2c_{66} - c_{22} + 2\bar{\beta}_2c_{12})l_1^2l_2^2 -\\
&\quad c_{66}\bar{\beta}_2^2 l_2^4 - c_{66}l_1^4].
\end{aligned}$$

Eqs. (8) using Eq. (7) are rewritten as

$$(U_1, U_2, U_3, U_4) = \sum_{q=1}^{8} (\bar{U}_{1q}, \bar{U}_{2q}, \bar{U}_{3q}, \bar{U}_{4q}) \mathrm{e}^{-\mathrm{i}\xi(l_3+\alpha_q)x_3}. \tag{13}$$

For each $\alpha_q, q = 1, 2, \ldots, 8$, using the Eqs. (9) and express the displacements ratios as

$$\begin{aligned} &\frac{D_1(\alpha_q)}{D(\alpha_q)} = \frac{\bar{U}_{2q}}{\bar{U}_{1q}} = \gamma_q, \qquad \frac{D_2(\alpha_q)}{D(\alpha_q)} = \frac{\bar{U}_{3q}}{\bar{U}_{1q}} = \delta_q, \\ &\frac{D_3(\alpha_q)}{D(\alpha_q)} = \frac{\bar{U}_{4q}}{\bar{U}_{1q}} = \Theta_q \quad \text{for} \quad q = 1, 2, \ldots, 8, \end{aligned} \tag{14}$$

where

$$\begin{aligned} D_1(\alpha_q) &= M_{23}(\alpha_q)M_{34}(\alpha_q)M_{41}(\alpha_q) + M_{24}(\alpha_q)M_{33}(\alpha_q)M_{41}(\alpha_q) - \\ &\quad M_{13}(\alpha_q)M_{24}(\alpha_q)M_{43}(\alpha_q) + M_{12}(\alpha_q)M_{34}(\alpha_q)M_{43}(\alpha_q) + \\ &\quad M_{13}(\alpha_q)M_{23}(\alpha_q)M_{44}(\alpha_q) - M_{12}(\alpha_q)M_{33}(\alpha_q)M_{44}(\alpha_q), \\ D_2(\alpha_q) &= M_{23}(\alpha_q q)M_{24}(\alpha_q)M_{41}(\alpha_q) + M_{12}(\alpha_q)M_{23}(\alpha_q)M_{44}(\alpha_q) + \\ &\quad M_{13}(\alpha_q)M_{24}(\alpha_q)M_{42}(\alpha_q) + M_{22}(\alpha_q)M_{34}(\alpha_q)M_{41}(\alpha_q) - \\ &\quad M_{13}(\alpha_q)M_{22}(\alpha_q)M_{44}(\alpha_q) - M_{12}(\alpha_q)M_{34}(\alpha_q)M_{42}(\alpha_q), \\ D_3(\alpha_q) &= M_{23}^2(\alpha_q)M_{41}(\alpha_q) - M_{22}(\alpha_q)M_{33}(\alpha_q)M_{41}(\alpha_q) - \\ &\quad M_{12}(\alpha_q)M_{23}(\alpha_q)M_{43}(\alpha_q) + M_{13}(\alpha_q)M_{22}(\alpha_q)M_{43}(\alpha_q) + \\ &\quad M_{12}(\alpha_q)M_{33}(\alpha_q)M_{42}(\alpha_q) - M_{13}(\alpha_q)M_{23}(\alpha_q)M_{42}(\alpha_q), \\ D(\alpha_q) &= M_{23}(\alpha_q)M_{34}(\alpha_q)M_{42}(\alpha_q) - M_{24}(\alpha_q)M_{33}(\alpha_q)M_{42}(\alpha_q) - \\ &\quad M_{22}(\alpha_q)M_{34}(\alpha_q q)M_{43}(\alpha_q) + M_{22}(\alpha_q)M_{33}(\alpha_q)M_{44}(\alpha_q) - \\ &\quad M_{23}^2(\alpha_q)M_{44}(\alpha_q) + M_{23}(\alpha_q)M_{24}(\alpha_q)M_{43}(\alpha_q). \end{aligned} \tag{15}$$

Then the solution given by Eq. (13) may be rewritten as

$$(U_1, U_2, U_3, U_4) = \sum_{q=1}^{8} (1, \gamma_q, \delta_q, \Theta_q) \bar{U}_{1q} \mathrm{e}^{-\mathrm{i}\xi(l_3+\alpha_q)x_3}. \tag{16}$$

In view of the continuity of the displacement components, temperature, tractions and temperature gradient across the interface of the two layers, the following conditions must be satisfied:

$$u^{\mathrm{I}}_{j_{x_3=0^-}} = u^{\mathrm{II}}_{j_{x_3=0^+}}, \qquad T^{\mathrm{I}}_{x_3=0^-} = T^{\mathrm{II}}_{x_3=0^+}, \tag{17}$$

$$\sigma^{\mathrm{I}}_{3j_{x_3=0^-}} = \sigma^{\mathrm{II}}_{3j_{x_3=0^+}}, \qquad T'^{\mathrm{I}}_{x_3=0^-} = T'^{\mathrm{II}}_{x_3=0^+}, \tag{18}$$

where $T' = \frac{\partial T}{\partial x_3}$ superscripts I and II refer to layers one and two respectively, 0^+ and 0^- are values of x_3 near zero. Because of periodicity of the deformation and thermoelastic stress fields, additional conditions obtained are

$$u^{\mathrm{I}}_{j_{x_3=h_1^-}} = u^{\mathrm{II}}_{j_{x_3=-h_2^+}}, \qquad T^{\mathrm{I}}_{x_3=h_1^-} = T^{\mathrm{II}}_{x_3=-h_2^+}, \tag{19}$$

$$\sigma^{\mathrm{I}}_{3j_{x_3=h_1^-}} = \sigma^{\mathrm{II}}_{3j_{x_3=-h_2^+}}, \qquad T'^{\mathrm{I}}_{x_3=h_1^-} = T'^{\mathrm{II}}_{x_3=-h_2^+}, \qquad j = 1, 2, 3. \tag{20}$$

On substituting the displacements, temperature, stresses and temperature gradient components into Eqs. (17)–(18), sixteen linear homogeneous equations for sixteen constants $U_{11}^{\mathrm{I}}, U_{12}^{\mathrm{I}}, \ldots$, U_{17}^{II} and U_{18}^{II} are obtained. For nontrivial solutions, the determinant of coefficient matrix must vanish. This yields the following characteristic equation:

$$\det\begin{pmatrix} P_{jk} & -\bar{P}_{jk} \\ Q_{jk} & -\bar{Q}_{jk} \end{pmatrix} = 0, \quad j,k = 1,2,\ldots,8. \tag{21}$$

The entries of 8×8 matrices $P_{jk}, \bar{P}_{jk}, Q_{jk}$ and $\bar{Q}_{jk}$ are

$$\begin{aligned}
&P_{1j} = 1, \quad P_{2j} = \gamma_j^{\mathrm{I}}, \quad P_{3j} = \delta_j^{\mathrm{I}}, \quad P_{4j} = \Theta_j^{\mathrm{I}},\\
&P_{5j} = b_{1j}^{\mathrm{I}} c_{55}^{\mathrm{I}}, \quad P_{6j} = b_{2j}^{\mathrm{I}} c_{44}^{\mathrm{I}}, \quad P_{7j} = b_{3j}^{\mathrm{I}}, \quad P_{8j} = -b_{4j}^{\mathrm{I}},\\
&\bar{P}_{1j} = 1, \quad \bar{P}_{2j} = \gamma_j^{\mathrm{II}}, \quad \bar{P}_{3j} = \delta_j^{\mathrm{II}}, \quad \bar{P}_{4j} = \Theta_j^{\mathrm{II}},\\
&\bar{P}_{5j} = \eta b_{1j}^{\mathrm{II}} c_{55}^{\mathrm{II}}, \quad \bar{P}_{6j} = \eta b_{2j}^{\mathrm{II}} c_{44}^{\mathrm{II}}, \quad \bar{P}_{7j} = \eta b_{3j}^{\mathrm{II}}, \quad \bar{P}_{8j} = \eta b_{4j}^{\mathrm{II}},\\
&Q_{jk} = P_{jk} E_k^-, \quad \bar{Q}_{jk} = \bar{P}_{jk} E_k^+,
\end{aligned} \tag{22}$$

where $E_j^- = \mathrm{e}^{-\mathrm{i}Q(l_3+\alpha_j^{(1)})h_1/h}$, $Q = \xi(h_1 + h_2)$, $E_j^+ = \mathrm{e}^{-\mathrm{i}Q(l_3+\alpha_j^{\mathrm{II}})h_2/h}$, $\eta = c_{11}^{\mathrm{II}}/c_{11}^{\mathrm{I}}$,

$$\begin{aligned}
b_{1j}^{(m)} &= l_1\delta_j^{(m)} - \alpha_j^{(m)}, \quad b_{2j}^{(m)} = l_2\delta_j^{(m)} - \alpha_j^{(m)}\gamma_j^{(m)},\\
b_{3j}^{(m)} &= \bar{c}_{13}^{(m)} l_1 + \bar{c}_{23}^{(m)} l_2 \gamma_j^{(m)} - \bar{c}_{33}^{(m)} \alpha_j^{(m)} \delta_j^{(m)} - \beta_3 \Theta_j^{(m)},\\
b_{4j}^{\mathrm{I}} &= (l_3 + \alpha_j^{(m)})\Theta_j^{(m)} = \mathrm{i}\xi\alpha_j^{(m)}\Theta_j^{(m)}, \quad \bar{c}_{jk}^{(m)} = c_{jk}^{(m)}/c_{11}^{(m)}, \quad m = \mathrm{I}, \mathrm{II}.
\end{aligned} \tag{23a}$$

From Eq. (21), we have $\det[P_{jk}]\det([-\bar{Q}_{jk}] - [Q_{jk}][P_{jk}]^{-1}[-\bar{P}_{jk}]) = 0$ which implies that

$$\text{either } \det[P_{jk}] = 0, \tag{23b}$$

$$\text{or } \det([-\bar{Q}_{jk}] - [Q_{jk}][P_{jk}]^{-1}[-\bar{P}_{jk}]) = 0. \tag{23c}$$

If Eq. (23b) holds true, then the problem reduces to a free wave propagation in a single thermoelastic plate of thickness h_1, and in this case $([-\bar{Q}_{jk}] - [Q_{jk}][P_{jk}]^{-1}[-\bar{P}_{jk}])$ will not exist as P_{jk}singular. On the hand P_{jk} is nonsingular $[P_{jk}]^{-1}$ exists and accordingly

$$\det([-\bar{Q}_{jk}] - [Q_{jk}][P_{jk}]^{-1}[-\bar{P}_{jk}]) = 0. \tag{24a}$$

Similarly Eq. (21) can also be written as

$$\det[-\bar{Q}_{jk}]\det([P_{jk}] - [-\bar{P}_{jk}][-\bar{Q}_{jk}]^{-1}[Q_{jk}]) = 0, \tag{24b}$$

which implies that either

$$\det[-\bar{Q}_{jk}] = 0, \tag{24c}$$

or

$$\det([P_{jk}] - [-\bar{P}_{jk}][-\bar{Q}_{jk}]^{-1}[Q_{jk}]) = 0. \tag{24d}$$

If Eq. (24b) holds true, then again the problem reduces to a single thermoelastic plate of thickness h_2, and $([-\bar{Q}_{jk}] - [Q_{jk}][P_{jk}]^{-1}[-\bar{P}_{jk}])$ will not exists as $\bar{Q}_{jk}$ is singular.

On the hand, if $\bar{Q}_{jk}$ is non-singular, therefore

$$\det([-\bar{Q}_{jk}] - [Q_{jk}][P_{jk}]^{-1}[-\bar{P}_{jk}]) = 0. \tag{25}$$

In order to solve the problem numerically it is sufficient to consider either Eq. (24a) or Eq. (25) for composite plates and to solve for free thermoelastic plate Eq. (23b) or Eq. (24b) can be considered.

4. Particular cases

4.1. Classical case

If the coupling constant $\varepsilon = 0$, then thermal and elastic fields decoupled from each other and from Eq. (11) we have $M_{41} = M_{42} = M_{43} = 0$. In this case Eq. (12) factorised into

$$(l_1^2 + \bar{K}_2 l_2^2 + \bar{K}_3 \alpha^2 - \omega^* \zeta^2 \tau)(\Delta \alpha^6 + F_1 \alpha^4 + F_2 \alpha^2 + F_3) = 0. \tag{26}$$

One of the factor of the above equation

$$\Delta \alpha^6 + F_1 \alpha^4 + F_2 \alpha^2 + F_3 = 0 \tag{27}$$

corresponds to the characteristic equation in the uncoupled thermoelasticity, where

$$\begin{aligned}
\Delta &= c_{33} c_{44} c_{55}, \\
F_1 &= [(c_{22} c_{33} - 2 c_{23} c_{44} - c_{23}^2) c_{55} + c_{33} c_{44} c_{66}] l_2^2 + \\
&\quad [(c_{33} - 2 c_{13} c_{55} - c_{13}^2) c_{44} + c_{33} c_{55} c_{66}] l_1^2 - \\
&\quad (c_{33} c_{44} + c_{33} c_{55} + c_{44} c_{55}) \zeta^2, \\
F_2 &= [(c_{33} - 2 c_{13} c_{55} - c_{13}^2) c_{66} + c_{44} c_{55}] l_1^4 + [(c_{22} c_{33} - 2 c_{23} c_{44} - c_{23}^2) c_{66} + c_{22} c_{55} c_{44}] l_2^4 + \\
&\quad [-c_{12}^2 c_{33} - 2(c_{33} c_{44} - c_{66} c_{23} c_{55} - c_{12} c_{44} c_{55} + c_{13} c_{22} c_{55} - 2 c_{44} c_{55} c_{66} - c_{13} c_{44} c_{66} + \\
&\quad c_{12} c_{33} c_{66} - c_{12} c_{13} c_{44} - c_{13} c_{23} c_{66} - c_{12} c_{23} c_{55} - c_{12} c_{13} c_{23}) - c_{13}^2 c_{22} + c_{22} c_{33} - c_{23}^2] l_1^2 l_2^2 + \\
&\quad [(2 c_{13} c_{55} - c_{66} c_{33} - c_{55} c_{44} - c_{44} - c_{33} - c_{66} c_{55} + c_{13}^2) l_1^2 + \\
&\quad (2 c_{23} c_{44} + c_{23}^2 - c_{22} c_{33} - c_{22} c_{55} - c_{66} c_{44} - c_{55} c_{44} - c_{33} c_{66}) l_2^2 + (c_{33} + c_{44} + c_{55}) \zeta^4] \zeta^2, \\
F_3 &= (c_{55} l_1^2 + c_{44} l_2^2 - \zeta^2) \{[(1 + c_{66}) l_1^2 + (c_{22} + c_{66}) l_2^2] \zeta^2 - \zeta^4 + \\
&\quad [(2 c_{22} c_{66} + c_{12}^2 - c_{22}) c_{55}] l_1^2 l_2^2 - c_{22} c_{66} l_2^4 - c_{66} l_1^4\}.
\end{aligned}$$

In this case, Eqs. (14) simplify to

$$\begin{aligned}
D_1(\alpha_q) &= M_{13}(\alpha_q) M_{23}(\alpha_q) - M_{12}(\alpha_q) M_{33}(\alpha_q), \\
D_2(\alpha_q) &= M_{12}(\alpha_q) M_{23}(\alpha_q) - M_{13}(\alpha_q) M_{22}(\alpha_q), \\
D_3(\alpha_q) &= 0, \qquad D(\alpha_q) = M_{22}(\alpha_q) M_{33}(\alpha_q) - M_{23}^2(\alpha_q)
\end{aligned} \tag{28}$$

and the reduced result corresponds to the purely elastic orthotropic materials, which is obtained and studied by Yamada and Nasser [29]. On the other hand, the second factor of the Eq. (26) is

$$l_1^2 + \bar{K}_2 l_2^2 + \bar{K}_3 \alpha^2 - \omega^* \zeta^2 \tau = 0, \tag{29}$$

which corresponds to the purely thermal wave. Hence thermal wave in the generalized theory of thermoelasticity is influenced by the thermal relaxation time τ.

4.2. Thermoelastic free waves

When layer I $=$ II and $h_1 = h_2 (\text{say}\, h)$ then the thickness of the layer is $2h$, on considering origin at mid of the plate, then the above analysis reduces to a single plate. In this case, the eight roots of Eq. (12) can be arranged in four pairs as $\alpha_{j+1} = -\alpha_j$, $j = 1, 3, 5, 7$.

It is observed from Eq. (11) that M_{13}, M_{23}, M_{34} and M_{43} are odd functions of α, and the other M_{ij}'s are even functions of α. On employing the thermal stresses and thermal gradient free surfaces conditions

$$\sigma_{3j} = T' = 0, \quad x_3 = \pm h, \quad j = 1, 2, 3, \tag{30}$$

and employing the relations (14), we have

$$\gamma_{q+1} = \gamma_q, \quad \delta_{q+1} = -\delta_q \text{ and } \Theta_{q+1} = \Theta_q. \tag{31}$$

Hence from (23a)

$$\begin{aligned} b_{1q+1} &= -b_{1q}, \quad b_{2q+1} = -b_{2q}, \quad b_{3q+1} = b_{3q} \text{ and } b_{4q+1} = -b_{4q}, && (32)\\ b_{1j} &= l_1\delta_j - \alpha_j, \quad b_{2j} = l_2\delta_j - \alpha_j\gamma_j, && \\ b_{3j} &= \bar{c}_{13}l_1 + \bar{c}_{23}l_2\gamma_j - \bar{c}_{33}\alpha_j\delta_j - \beta_3\Theta_j, \quad b_{4j} = -\mathrm{i}\xi\alpha_j\Theta_j\bar{c}_{jk} = c_{jk}/c_{11}, && (33)\\ \det\left[P_{jk}\right] &= 0. && (34) \end{aligned}$$

Eq. (34) is the corresponding characteristic equation for free waves in generalized thermoelasticity. Further, if thickness $d = (h_1 + h_2) \to \infty$, in Eq. (34) then the problem reduces to thermoelastic surface waves.

4.3. *Coupled thermoelasticity*

This is the case, when thermal relaxation times $\tau_0 = \tau_1 = 0$ and hence, $\tau = \tau_g = \mathrm{i}/\omega$. Following above, we arrived at frequency equation of the coupled thermoelasticity. When $\tau_1 = \tau_0 \neq 0$, characteristic Eq. (21) becomes the frequency equation in the LS theory of generalized thermoelasticity.

5. Numerical results and discussion

Using Eq. (24a) numerical results are presented to exhibit the dependence of dispersion on the angle of propagation and thermal relaxation time. The materials chosen for this purpose are aluminum epoxy composite as layer I ($h_1 = 0.6$) and carbon steel as layer II ($h_2 = 0.4$).

Since the distinction among the wave mode types of thermoelastic waves in anisotropic plates is somewhat artificial, as the thermal and elastic wave modes are generally coupled, they are referred to as quasilongitudinal and quasitransverse, quasishear horizontal modes and quasithermal. For wave propagation in the direction of higher symmetry (see Section 4), some wave types revert to pure modes and lead to a simple characteristic equation of lower order, and consequently to the loss of pure wave modes in the direction of general propagation. Here Fig. 2 depicts the dispersion curves for the direction cosines of propagation $l_1 = 0.259$, $l_2 = 0.542$, and $l_3 = 0.799$, whereas Fig. 3 demonstrate the dispersion behavior when the direction cosines of propagation are same but the coupling constant $\varepsilon = 0$, i.e., thermal and elastic fields are not coupled.

Similarly, dispersion curves with the direction cosines of propagation $l_1 = 0.195$, $l_2 = 0.515$, and $l_3 = 0.834$ are shown in Fig. 4, whereas when the direction cosines of propagation are same but the coupling constant $\varepsilon = 0$.

Similarly, on considering the direction cosines of propagation $l_1 = 0.125$, $l_2 = 0.707$, and $l_3 = 0.696$ dispersion curves are shown in Fig. 6, whereas when the coupling constant $\varepsilon = 0$, keeping the same direction cosines dispersion curves are shown in Fig. 7.

It is observed that in generalized thermoelasticity, at zero wave number limits, each figure (Figs. 2, 4 and 6) displays four thermoelastic wave speeds corresponding to one quasilongitudinal, two quasitransverse and one quasithermal. It is apparent that the largest value corresponds to the quasi-longitudinal and the additional mode appears is a quasi-thermal mode. At low wave number limits, modes are found to highly influenced and also vary with the direction. A small

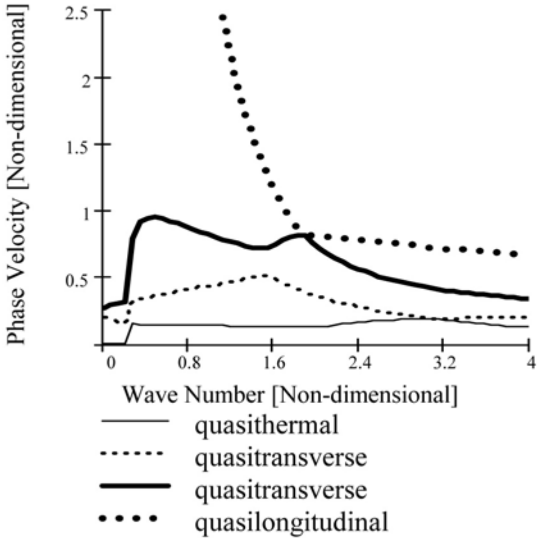

Fig. 2. Phase velocity versus wave number for the direction cosine $l_1 = 0.259$, $l_2 = 0.542$ and $l_3 = 0.799$ in generalized thermoelasticity

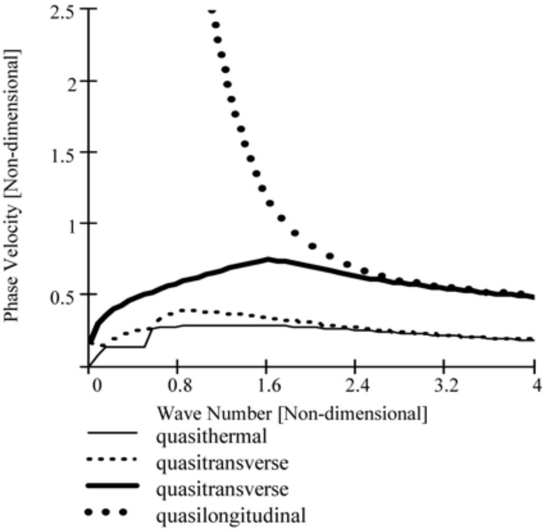

Fig. 3. Phase velocity versus wave number for the direction cosine $l_1 = 0.259$, $l_2 = 0.542$ and $l_3 = 0.799$ when the coupling parameter is zero

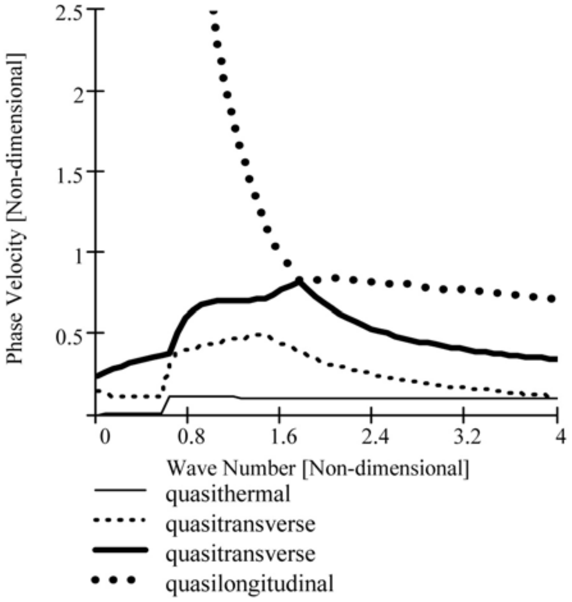

Fig. 4. Phase velocity versus wave number for the direction cosine $l_1 = 0.195$, $l_2 = 0.515$ and $l_3 = 0.834$ in generalized thermoelasticity

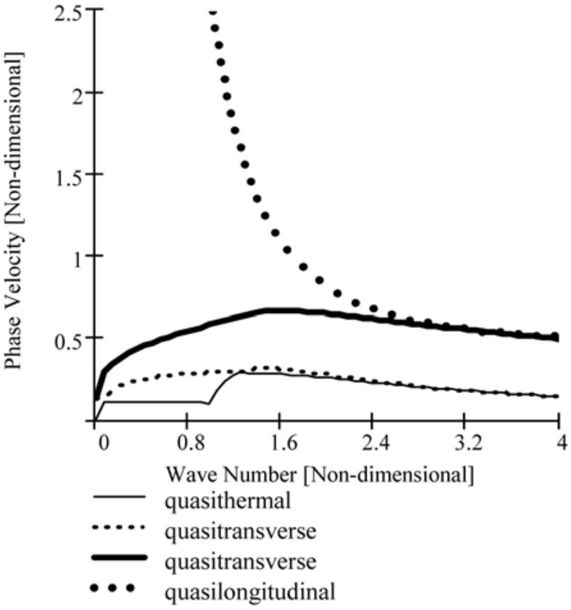

Fig. 5. Phase velocity versus wave number for the direction cosine $l_1 = 0.195$, $l_2 = 0.515$ and $l_3 = 0.834$ when the coupling parameter is zero

change is observed in these modes values as ξ increases and others higher modes appear, one of the modes seems to be associated with quick change in the slope. It is also observed that with change in direction, lower modes appear to have large influence than the higher modes where a small variation is noticed. When the when the coupling constant $\varepsilon = 0$, i.e., thermal and elastic fields are not coupled, Figs. 3, 5 and 7 demonstrate the dispersion behavior of wave speed modes with different angles of propagation. From these figures, it is observed that at low wave number limits, although wave speed modes are dispersive, but are different from the coupled case. Thus in generalized thermoelasticity, at low values of the wave number, only

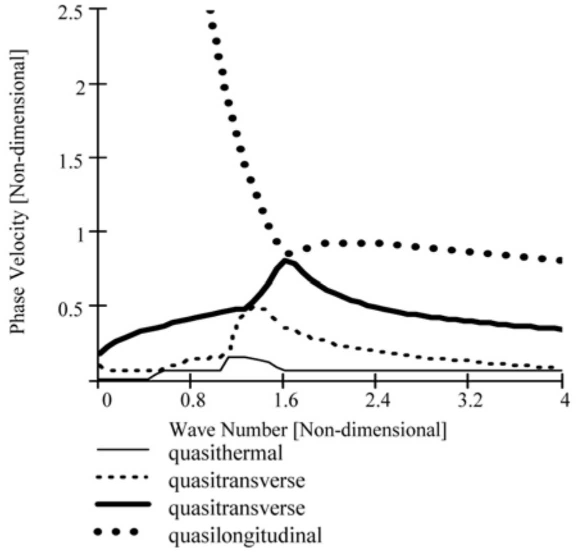

Fig. 6. Phase velocity versus wave number with direction cosine $l_1 = 0.125$, $l_2 = 0.707$ and $l_3 = 0.696$ in generalized thermoelasticity

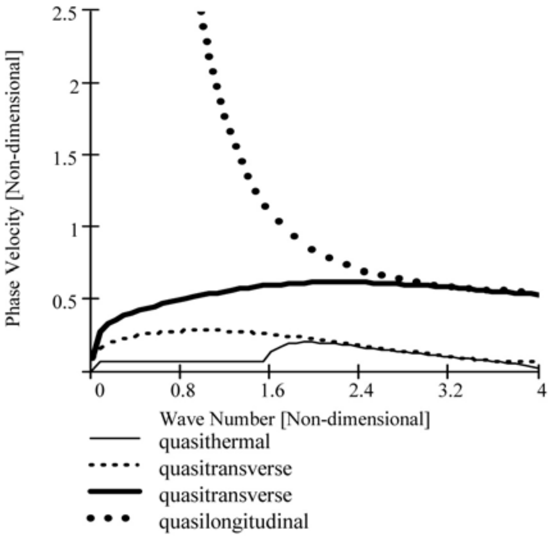

Fig. 7. Phase velocity versus wave number for the direction cosine $l_1 = 0.125$, $l_2 = 0.707$ and $l_3 = 0.696$ when the coupling parameter is zero

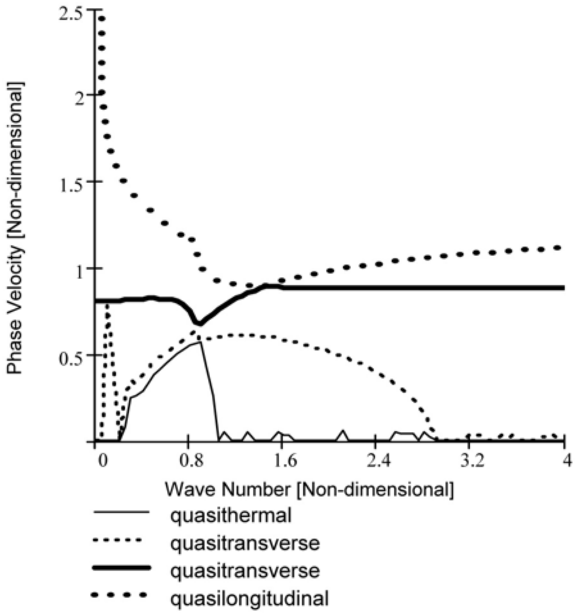

Fig. 8. Phase velocity versus wave number in GL theory of thermoelasticity with thermal relaxation times $\tau_0 = 2 \cdot 10^{-7}$, $\tau_1 = 2 \cdot 10^{-6}$

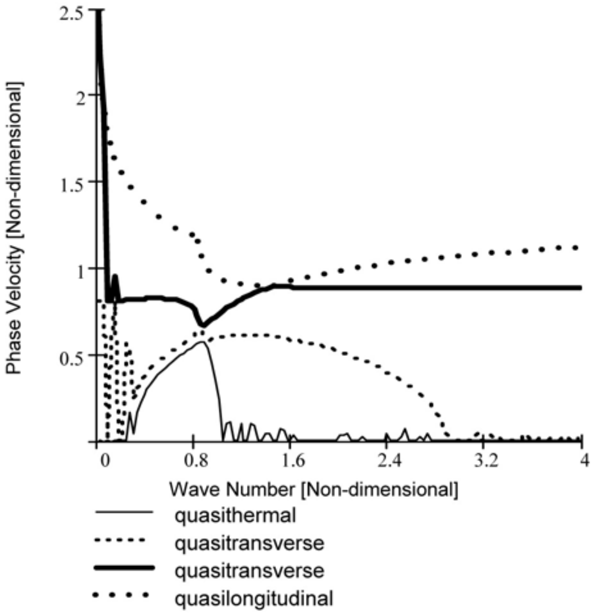

Fig. 9. Phase velocity versus wave number in GL theory of thermoelasticity with thermal relaxation times $\tau_0 = 2 \cdot 10^{-7}$, $\tau_1 = 10 \cdot 10^{-7}$

the lower modes hihgly affected and the little change is observed at the relatively high values of wave number. The low value region of the wave number is found to be of more physical interest in generalized thermoelasticity. As high wave number limits exhibit no effect on wave speeds, therefore the second sound effects are short lived in the laminated composites plates in generalized thermoelasticity.

To observe the influence of the thermal relaxations, selected values of thermal relaxation times τ_1 and τ_0 are cosidered, Figs. 8–10 demonstarte the variations of phase velocity with wave number and the dispersive character of quasilongitudinal, quasitransverse and quasither-

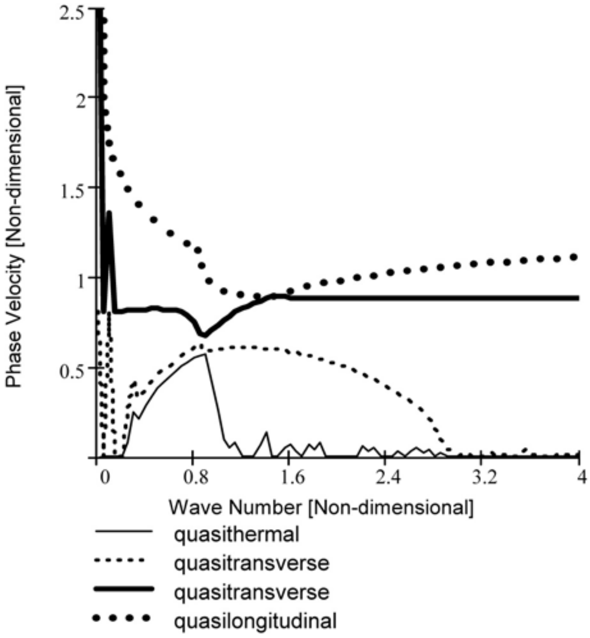

Fig. 10. Phase velocity versus wave number in GL theory of thermoelasticity with thermal relaxation times $\tau_0 = 2 \cdot 10^{-7}$, $\tau_1 = 4 \cdot 10^{-7}$

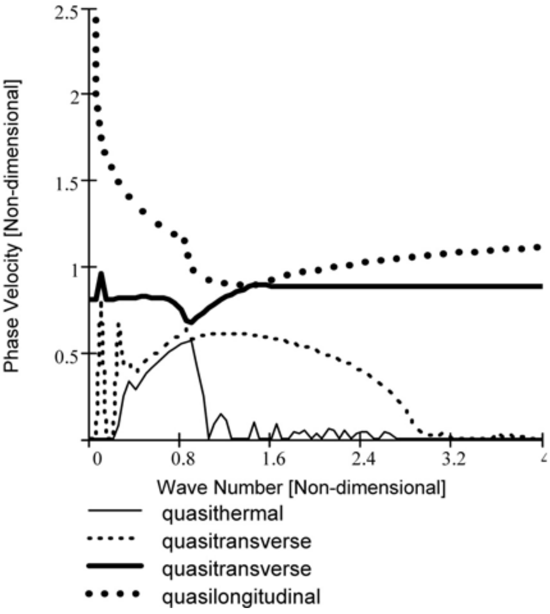

Fig. 11. Phase velocity versus wave number in LS theory of thermoelasticity with thermal relaxation times $\tau_0 = 2 \cdot 10^{-7}$

mal modes are represented. Quasilongitudinal, quasitransverse (two) and quasi-thermal waves are found coupled with each other due to the thermal and anisotropic effects, also wave-like behavior of the quasi-thermal modes is characterized in Green and Lindsay (GL) thermoelasticity theory. Also Fig. 11 is drawn by considering τ_0 only, a single time constant which represents the dispersion curve in Lord and Shulman (LS) theory.

Although the thermal relaxation times τ_1 and τ_0 are derived from distinctively different physical assumptions and physical laws, the dispersion behavior described by LS and GL theory for thermoelastic waves are remarkably similar even in laminated composites plates. It is probably due to the fact that even though the theories are entirely different in their approach to form a coupled thermoelasticity theory, they are remarkably similar in their formulation.

6. Conclusion

Dispersion of a 3D layered heat conducting composite plate in an arbitrary direction in the theory of generalized thermoelasticity is studied. Equations of motion for 3D continuum formulated for an infinite layered plate of an anisotropic thermoelastic medium with uniformly distributed temperature. The Floquet method is used for the derivation of general solution of displacements and temperature distributions. Special cases such as classical, free waves and coupled thermoelasticity are also presented and discussed. Influence of wave propagation direction on plate dispersion is analysed numerically and analytically.

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