Thermoelastic wave propagation in laminated composites plates

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Abstract

The dispersion of thermoelastic waves propagation in an arbitrary direction in laminated composites plates is studied in the framework of generalized thermoelasticity in this article. Three dimensional field equations of thermoelasticity with relaxation times are considered. Characteristic equation is obtained on employing the continuity of displacements, temperature, stresses and thermal gradient at the layers’ interfaces. Some important particular cases such as of free waves on reducing plates to single layer and the surface waves when thickness tends to infinity are also discussed. Uncoupled and coupled thermoelasticity are the particular cases of the obtained results. Numerical results are also obtained and represented graphically.

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1. Introduction

Increasing use of advanced composites as important structural components in modern high speed aircraft, missile, marine vehicles, and other aerospace structures, and various other applications has led to widespread research activities in the field of composite materials. Composites consist of different materials, so they are inhomogeneous and anisotropic. Different mechanical and thermal properties between constituents of such composites structures, like temperature changes, can generate residual stresses, which may lead to interface de-bonding. A possible failure of the system has intensified the need to study the thermoelastic wave propagation, especially in the form of precise numerical calculations. Consequently, it is of interest to investigate the feasibility of nondestructively, monitoring thermal, mechanical and aging in composites.

Extensive review on the dynamic behavior of anisotropic plate theories can be found in [1] and [14] and problems of wave propagation in periodically layered anisotropic media have been considered and studied in [16,28] and [3]. Dynamic behavior of the problems on the theories of laminated and composite plates have been investigated by authors [12] and [18–23]. Reasonable number of investigations of such advanced materials and their analysis also have been reported in [10,19]. In [15] a transfer matrix technique to obtain the dispersion relation curves of elastic waves propagating in multilayered anisotropic media i.e., composite laminate is developed and detailed review on the wave propagation in layered anisotropic media is studied in [11]. In [9], general problem of thermoelastic waves in anisotropic periodically laminated composites in thermoelasticity is studied.

Theory of thermoelasticity is well established, one can see the works in references [17] and [5]. Literature in this field is rather large to account for the phenomena involving the finite propagation velocity of the thermal wave, and can confer with the reference [4]. These modified

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coupled theories of thermoelasticity are based on hyperbolic-type equations for temperature and are closely connected with the theories of second sound, which consider heat propagation as a wavelike phenomenon. In the literature, addressing linear theories with relaxation time, most attention is given to the models formulated in [13] and [8]. Theory in [13] called Lord and Shulman (LS) theory is based on a modified Fourier’s Law of heat conduction with one relaxation time to dictate the relaxation of thermal propagation, as well as the rate of change of strain rate and the rate of change of heat generation. Green and Lindsay (GL) theory is based on a rigorous treatment of thermodynamics, and a form of the entropy inequality. The literature dedicated to hyperbolic thermoelastic models is quite large and its detailed review can be found in [6, 7].

Theory of generalized thermoelasticity [13] is extended to anisotropic heat conducting elastic materials by [2], and hence it is valid for both isotropic and anisotropic bodies. The propagation of harmonic waves in a laminated composite plate consisting of an arbitrary number of layers is studied in [9]. Various problems of infinite plates in the context of generalized theories thermoelasticity and the propagation of waves in layered anisotropic media in generalized thermoelasticity is investigated [24–27]. Yamada and Nasser [29] have studied harmonic wave’s propagation direction in orthotropic composites.

In this article propagation of thermoelastic waves in layered laminated composites, where the direction of the corresponding harmonic waves makes an arbitrary angle with respect to the layers is examined in the context of generalized thermoelasticity with two thermal relaxation times. Three dimensional field equations of thermoelasticity are considered for this study and the corresponding characteristic equation is obtained on employing the continuity of displacements, temperature, thermal stresses and thermal gradient at the layers’ interface. Some important particular cases such as of free waves on reducing plates to single layer and the surface waves when thickness tends to infinity are also discussed. Numerical results are also obtained and represented graphically.

2. Formulation

Consider a set of Cartesian coordinate system \( x_i = (x_1, x_2, x_3) \) in such a manner that \( x_3 \)-axis is normal to the layering. The basic field equations of generalized thermoelasticity for an infinite generally anisotropic thermoelastic medium at uniform temperature \( T_0 \) in the absence of body forces and heat sources are

\[
\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial^2 u_i}{\partial t^2},
\]

\[
K_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} - \rho C_e \left( \frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} \right) = T_0 \beta_{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial t} \right).
\]

Constitutive relations for anisotropic materials in the context of generalized thermoelasticity are following:

\[
\sigma_{ij} = c_{ijkl} e_{kl} - \beta_{ij} (T + \tau_1 \dot{T}),
\]

\[
\beta_{ij} = c_{ijkl} \alpha_{kl}, \quad i, j, k, l = 1, 2, 3,
\]

where \( \rho \) is the density of the \( n \)th layer, \( t \) is time, \( u_i \) is the displacement in the \( x_i \) direction, \( K_{ij} \) are the thermal conductivities, \( \sigma_{ij} \) and \( e_{ij} \) are the stress and strain tensor respectively, \( C_e \) is the specific heat at constant strain, \( \beta_{ij} \) are thermal moduli, \( \alpha_{ij} \) is the thermal expansion tensor, \( T \)
is temperature, and $c_{ijkl}$ is the fourth order tensor of the elasticity. The quantities $c_{ijkl}$, $\alpha_{ij}$, $\beta_{ij}$ satisfy the symmetry conditions

$$c_{ijkl} = c_{klij} = c_{ijlk} = c_{jikl}, \quad \alpha_{ij} = \alpha_{ji}, \quad \beta_{ij} = \beta_{ji}. \quad (5)$$

The parameter $\tau_1$ and $\tau_0$ are the thermal-mechanical relaxation time and the thermal relaxation time of the GL theory and satisfy the inequality $\tau_1 \geq \tau_0 \geq 0$. Strain-displacement relation is

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (6)$$

In addition, at the interface between two layers the tractions, temperature gradient, displacements and temperature must be continuous.

### 3. Analysis

For harmonic waves propagating in an arbitrary direction, the displacements components $u_1$, $u_2$, $u_3$ and temperature $T$ are written as

$$(u_1, u_2, u_3, T) = \{ U_1(x_3), U_2(x_3), U_3(x_3), U_4(x_3) \} e^{i(\xi l_1 x_1 + l_2 x_2 + l_3 x_3 - ct)}, \quad (7)$$

where $\xi$ is the wave number, $c = \omega/\xi$ is the phase velocity, $i = \sqrt{-1}$, $\omega$ is the circular frequency, $l_1, l_2$ and $l_3$ are the direction cosine defining the propagation direction as in Fig. 1.

![Fig. 1. Two-phase orthotropic layered thermoelastic composite plate. The direction of the propagation vector are denoted as $l_1, l_2$ and $l_3$](image)

$U_j$ and $T$ are the constants related to the amplitudes of displacement and temperature, Floquet’s theory requires functions $U_j$ ($j = 1, 2, 3$ and $4$) to have the same periodicity as the layering. Hence the problem is reduced to that of one pair of layers, where

$$U_j = \bar{U}_j e^{-i(\xi l_3 + \alpha)x_3}, \quad j = 1, 2, 3, 4, \quad (8)$$

where $\bar{U}_j$ are constants. On substitution of Eq. (8) into Eqs. (1)–(2), via (3)–(6) and specializing the equations for orthotropic media, it follows that

$$M_{mn}(\alpha) \bar{U}_n = 0, \quad m, n = 1, 2, 3, 4, \quad (9)$$
where

\[ M_{11} = (l_1^2 + l_2^2 \bar{c}_{66} + \alpha^2 \bar{c}_{55} - \zeta^2), \quad M_{12} = (\bar{c}_{12} + \bar{c}_{66}) l_1 l_2, \]  
\[ M_{13} = - (\bar{c}_{13} + \bar{c}_{55}) l_1 \alpha, \quad M_{14} = l_1, \]  
\[ M_{22} = (l_1^2 \bar{c}_{66} + l_2^2 \bar{c}_{22} + \alpha^2 \bar{c}_{44} - \zeta^2), \quad M_{23} = -(\bar{c}_{23} + \bar{c}_{44}) l_2 \alpha, \quad M_{24} = \beta_2 l_2, \]  
\[ M_{33} = (l_1^2 \bar{c}_{55} + l_2^2 \bar{c}_{33} + \alpha^2 \bar{c}_{33} - \zeta^2), \quad M_{34} = - \beta_3 \alpha, \quad M_{41} = \varepsilon \omega^* \zeta^2 l_1 \tau_g, \]  
\[ M_{42} = \varepsilon \omega^* \zeta^2 l_2 \beta_2 \tau_g, \quad M_{43} = - \varepsilon \omega^* \zeta^2 \alpha \beta_3 \tau_g, \quad M_{44} = l_1^2 + \bar{K}_2 l_2^2 + \bar{K}_3 \alpha^2 - \omega^* \zeta^2 \tau, \]  

where \( \zeta^2 = \frac{\varepsilon^2}{\kappa_t^2}, \omega^* = \frac{\kappa_t \Gamma_0}{\rho \varepsilon \kappa_{11}}, \) and \( \tau_g = \tau_1 + i/\omega, \tau = \tau_0 + i \omega. \) The existence of nontrivial solutions for \( U_j \) \( (j = 1, 2, 3, 4) \) demands the vanishing of the determinant in Eqs. (9), and yields the eighth degree polynomial equation

\[ \alpha^8 + A_1 \alpha^6 + A_2 \alpha^4 + A_3 \alpha^2 + A_4 = 0, \]  

where the coefficients \( A_1, A_2, A_3 \) and \( A_4 \) are

\[ A_1 = \frac{[Q_1 \omega^* \varepsilon \tau_g \zeta^2 + P_1 \bar{K}_3 + c_{33} c_{44} c_{55} (l_1^2 + l_2^2 \bar{K}_2 - \omega^* \tau \zeta^2)]}{\Delta}, \]  
\[ A_2 = \frac{[Q_2 \omega^* \varepsilon \tau_g \zeta^2 + P_2 \bar{K}_3 + P_1 (l_1^2 + \bar{K}_2 l_2^2 - \omega^* \tau \zeta^2)]}{\Delta}, \]  
\[ A_3 = \frac{[Q_3 \omega^* \varepsilon \tau_g \zeta^2 + P_3 \bar{K}_3 + P_2 (l_1^2 + l_2^2 \bar{K}_2 - \omega^* \tau \zeta^2)]}{\Delta}, \]  
\[ A_4 = \frac{[Q_4 \omega^* \varepsilon \tau_g \zeta^2 + P_3 (l_1^2 + l_2^2 \bar{K}_2 - \omega^* \tau \zeta^2)]}{\Delta}. \]  

The image text contains mathematical equations and expressions, with each equation and expression meticulously formatted to ensure clarity and readability. The text is a continuation of a mathematical analysis, likely related to the study of polynomials and their solutions. The notation and symbols used are consistent with mathematical conventions, indicating a formal and rigorous approach to the subject matter. The presence of Greek letters, such as \( \alpha, \beta, \gamma, \) and \( \lambda, \) as well as constants and variables, suggests a high level of abstraction typical of advanced mathematical research or educational materials. The text is dense and requires a strong background in mathematics to fully understand and apply the concepts presented. The equations appear to be derived from a specific context, possibly related to physics, engineering, or another field that employs similar mathematical frameworks. The notation and terminology used are essential for accurately interpreting the results and applying the findings to practical or theoretical problems.
Eqs. (8) using Eq. (7) are rewritten as
\[
(U_1, U_2, U_3, U_4) = \sum_{q=1}^{8} (\tilde{U}_{1q}, \tilde{U}_{2q}, \tilde{U}_{3q}, \tilde{U}_{4q}) e^{-i\xi(l_3+\alpha_q)x_3}.
\] (13)

For each \(\alpha_q, q = 1, 2, \ldots, 8\), using the Eqs. (9) and express the displacements ratios as
\[
\frac{D_1(\alpha_q)}{D(\alpha_q)} = \frac{\tilde{U}_{2q}}{\tilde{U}_{1q}} = \gamma_q, \quad \frac{D_2(\alpha_q)}{D(\alpha_q)} = \frac{\tilde{U}_{3q}}{\tilde{U}_{1q}} = \delta_q,
\]
\[
\frac{D_3(\alpha_q)}{D(\alpha_q)} = \frac{\tilde{U}_{4q}}{\tilde{U}_{1q}} = \Theta_q \quad \text{for} \quad q = 1, 2, \ldots, 8,
\]

where
\[
D_1(\alpha_q) = M_{23}(\alpha_q)M_{34}(\alpha_q)M_{41}(\alpha_q) + M_{24}(\alpha_q)M_{33}(\alpha_q)M_{41}(\alpha_q) - M_{13}(\alpha_q)M_{23}(\alpha_q)M_{43}(\alpha_q) + M_{12}(\alpha_q)M_{34}(\alpha_q)M_{43}(\alpha_q) + M_{13}(\alpha_q)M_{23}(\alpha_q)M_{44}(\alpha_q) - M_{12}(\alpha_q)M_{33}(\alpha_q)M_{44}(\alpha_q),
\]
\[
D_2(\alpha_q) = M_{23}(\alpha_q)M_{24}(\alpha_q)M_{41}(\alpha_q) + M_{12}(\alpha_q)M_{23}(\alpha_q)M_{44}(\alpha_q) + M_{13}(\alpha_q)M_{23}(\alpha_q)M_{42}(\alpha_q) + M_{22}(\alpha_q)M_{34}(\alpha_q)M_{41}(\alpha_q) - M_{13}(\alpha_q)M_{22}(\alpha_q)M_{44}(\alpha_q) - M_{12}(\alpha_q)M_{33}(\alpha_q)M_{42}(\alpha_q),
\]
\[
D_3(\alpha_q) = M_{23}(\alpha_q)M_{41}(\alpha_q) - M_{22}(\alpha_q)M_{33}(\alpha_q)M_{41}(\alpha_q) - M_{12}(\alpha_q)M_{23}(\alpha_q)M_{43}(\alpha_q) + M_{13}(\alpha_q)M_{22}(\alpha_q)M_{43}(\alpha_q) + M_{12}(\alpha_q)M_{33}(\alpha_q)M_{42}(\alpha_q) - M_{13}(\alpha_q)M_{23}(\alpha_q)M_{42}(\alpha_q),
\]
\[
D(\alpha_q) = M_{23}(\alpha_q)M_{34}(\alpha_q)M_{42}(\alpha_q) - M_{24}(\alpha_q)M_{33}(\alpha_q)M_{42}(\alpha_q) - M_{22}(\alpha_q)M_{34}(\alpha_q)M_{43}(\alpha_q) + M_{22}(\alpha_q)M_{33}(\alpha_q)M_{44}(\alpha_q) - M_{23}(\alpha_q)M_{44}(\alpha_q) + M_{23}(\alpha_q)M_{24}(\alpha_q)M_{43}(\alpha_q).
\]

Then the solution given by Eq. (13) may be rewritten as
\[
(U_1, U_2, U_3, U_4) = \sum_{q=1}^{8} (1, \gamma_q, \delta_q, \Theta_q)\tilde{U}_{1q} e^{-i\xi(l_3+\alpha_q)x_3}.
\] (16)

In view of the continuity of the displacement components, temperature, tractions and temperature gradient across the interface of the two layers, the following conditions must be satisfied:
\[
u_{I,j_{x_3=0}^+}^{1} = u_{II,j_{x_3=0}^+}^{1}, \quad T_{x_3=0}^{I} = T_{x_3=0}^{II}, \quad \sigma_{I,j_{x_3=0}^+}^{3} = \sigma_{II,j_{x_3=0}^+}^{3}, \quad T_{x_3=0}^{I} = T_{x_3=0}^{II},
\]
\[
u_{I,j_{x_3=0}^-}^{1} = u_{II,j_{x_3=0}^-}^{1}, \quad T_{x_3=0}^{I} = T_{x_3=0}^{II}, \quad \sigma_{I,j_{x_3=0}^-}^{3} = \sigma_{II,j_{x_3=0}^-}^{3}, \quad T_{x_3=0}^{I} = T_{x_3=0}^{II},
\]
where \(T' = \frac{\partial T}{\partial x_3}\) superscripts I and II refer to layers one and two respectively, \(0^+\) and \(0^-\) are values of \(x_3\) near zero. Because of periodicity of the deformation and thermoelastic stress fields, additional conditions obtained are
\[
u_{I,j_{x_3=h_1}^+}^{1} = u_{II,j_{x_3=-h_2}^+}^{1}, \quad T_{x_3=h_1}^{I} = T_{x_3=-h_2}^{II}, \quad \sigma_{I,j_{x_3=h_1}^+}^{3} = \sigma_{II,j_{x_3=-h_2}^+}^{3}, \quad T_{x_3=h_1}^{I} = T_{x_3=-h_2}^{II},
\]
\[
u_{I,j_{x_3=h_1}^-}^{1} = u_{II,j_{x_3=-h_2}^-}^{1}, \quad T_{x_3=h_1}^{I} = T_{x_3=-h_2}^{II}, \quad \sigma_{I,j_{x_3=h_1}^-}^{3} = \sigma_{II,j_{x_3=-h_2}^-}^{3}, \quad T_{x_3=h_1}^{I} = T_{x_3=-h_2}^{II}, \quad j = 1, 2, 3.
\] (19) (20)
On substituting the displacements, temperature, stresses and temperature gradient components into Eqs. (17)–(18), sixteen linear homogeneous equations for sixteen constants \( U_{11}^1, U_{12}^1, \ldots, U_{11}^{12} \) and \( T_{11}^{12} \) are obtained. For nontrivial solutions, the determinant of coefficient matrix must vanish. This yields the following characteristic equation:

\[
\det \left( \begin{array}{cc}
P_{jk} & -\hat{P}_{jk} \\
Q_{jk} & -\hat{Q}_{jk}
\end{array} \right) = 0, \quad j, k = 1, 2, \ldots, 8.
\] (21)

The entries of \( 8 \times 8 \) matrices \( P_{jk}, \hat{P}_{jk}, Q_{jk} \) and \( \hat{Q}_{jk} \) are

\[
\begin{align*}
P_{1j} &= 1, \quad P_{2j} = \gamma_j^1, \quad P_{3j} = \delta_j^1, \quad P_{4j} = \Theta_j^1, \\
P_{5j} &= b_{1j}^1 c_{55}, \quad P_{6j} = b_{1j}^1 c_{44}, \quad P_{7j} = b_{2j}^1, \quad P_{8j} = -b_{4j}^1, \\
\hat{P}_{1j} &= 1, \quad \hat{P}_{2j} = \gamma_j^II, \quad \hat{P}_{3j} = \delta_j^II, \quad \hat{P}_{4j} = \Theta_j^II, \\
\hat{P}_{5j} &= \eta b_1^{1II} P_{55}, \quad \hat{P}_{6j} = \eta b_2^{1II} P_{44}, \quad \hat{P}_{7j} = \eta b_3^{1II}, \quad \hat{P}_{8j} = \eta b_4^{1II}, \\
Q_{jk} &= P_{jk} E_k^+, \quad \hat{Q}_{jk} = \hat{P}_{jk} E_k^+,
\end{align*}
\] (22)

where \( E_j^+ = e^{-iQ(l_3 + \alpha_j^1)h_1/h}, Q = \xi (h_1 + h_2), E_j^- = e^{-iQ(l_3 + \alpha_j^II)h_1/h}, \eta = c_{11}^II/c_{11}^I, \)

\[
\begin{align*}
b_{1j}^{(m)} &= l_1 \delta_j^{(m)} - \alpha_j^{(m)}, \quad b_{2j}^{(m)} = l_2 \delta_j^{(m)} - \alpha_j^{(m)} \gamma_j^{(m)}, \\
b_{3j}^{(m)} &= c_{13}^{(m)} l_1 + c_{23}^{(m)} l_2 \gamma_j^{(m)} - c_{33}^{(m)} \alpha_j^{(m)} \delta_j^{(m)} - \beta_3^{(m)} \Theta_j^{(m)}, \\
b_{4j}^{(m)} &= (l_3 + \alpha_j^{(m)}) \Theta_j^{(m)} = i \xi \alpha_j^{(m)} \Theta_j^{(m)}, \quad c_{4j}^{(m)} = c_{4j}^{(m)} / c_{11}^{II}, \quad m = I, II.
\end{align*}
\] (23a)

From Eq. (21), we have \( \det [P_{jk}] \det ([-\hat{Q}_{jk}] - [Q_{jk}] [P_{jk}]^{-1} [-\hat{P}_{jk}] = 0 \) which implies that

\[
\text{either } \det [P_{jk}] = 0, \quad \text{(23b)}
\]

\[
\text{or } \det ([-\hat{Q}_{jk}] - [Q_{jk}] [P_{jk}]^{-1} [-\hat{P}_{jk}] = 0. \quad \text{(23c)}
\]

If Eq. (23b) holds true, then the problem reduces to a free wave propagation in a single thermoelastic plate of thickness \( h_1 \), and in this case \(([-\hat{Q}_{jk}] - [Q_{jk}] [P_{jk}]^{-1} [-\hat{P}_{jk}]) \) will not exist as \( P_{jk} \) singular. On the hand \( P_{jk} \) is nonsingular \([P_{jk}]^{-1} \) exists and accordingly

\[
\det ([-\hat{Q}_{jk}] - [Q_{jk}] [P_{jk}]^{-1} [-\hat{P}_{jk}] = 0. \quad \text{(24a)}
\]

Similarly Eq. (21) can also be written as

\[
\det [-\hat{Q}_{jk}] \det ([P_{jk}] - [-\hat{P}_{jk}] [-\hat{Q}_{jk}]^{-1} [Q_{jk}] = 0, \quad \text{(24b)}
\]

which implies that either

\[
\det [-\hat{Q}_{jk}] = 0, \quad \text{(24c)}
\]

or

\[
\det ([P_{jk}] - [-\hat{P}_{jk}] [-\hat{Q}_{jk}]^{-1} [Q_{jk}] = 0. \quad \text{(24d)}
\]

If Eq. (24b) holds true, then again the problem reduces to a single thermoelastic plate of thickness \( h_2 \), and \(([-\hat{Q}_{jk}] - [Q_{jk}] [P_{jk}]^{-1} [-\hat{P}_{jk}]) \) will not exists as \( \hat{Q}_{jk} \) is singular. On the hand, if \( \hat{Q}_{jk} \) is non-singular, therefore

\[
\det ([-\hat{Q}_{jk}] - [Q_{jk}] [P_{jk}]^{-1} [-\hat{P}_{jk}] = 0. \quad \text{(25)}
\]

In order to solve the problem numerically it is sufficient to consider either Eq. (24a) or Eq. (25) for composite plates and to solve for free thermoelastic plate Eq. (23b) or Eq. (24b) can be considered.
4. Particular cases

4.1. Classical case

If the coupling constant \( \varepsilon = 0 \), then thermal and elastic fields decoupled from each other and from Eq. (11) we have \( M_{41} = M_{42} = M_{43} = 0 \). In this case Eq. (12) factorised into

\[
(l_1^2 + \tilde{K}_3 l_2^2 + \tilde{K}_3 \alpha^2 - \omega^* \zeta^2 \tau)(\Delta \alpha^6 + F_1 \alpha^4 + F_2 \alpha^2 + F_3) = 0.
\]

One of the factor of the above equation

\[
\Delta \alpha^6 + F_1 \alpha^4 + F_2 \alpha^2 + F_3 = 0
\]

corresponds to the characteristic equation in the uncoupled thermoelasticity, where

\[
\Delta = c_{33} c_{44} c_{55},
\]

\[
F_1 = [(c_{22} c_{33} - 2 c_{23} c_{44} - c_{23}^2) c_{55} + c_{33} c_{44} c_{66}] l_2^2 + [(c_{33} - 2 c_{13} c_{55} - c_{13}^2) c_{44} + c_{33} c_{55} c_{66}] l_1^2 - (c_{33} c_{44} + c_{33} c_{55} + c_{44} c_{55}) \zeta^2,
\]

\[
F_2 = [(c_{33} - 2 c_{13} c_{55} - c_{13}^2) c_{66} + c_{44} c_{55}] l_1^4 + [(c_{22} c_{33} - 2 c_{23} c_{44} - c_{23}^2) c_{66} + c_{22} c_{55} c_{44}] l_2^4 + [-c_{12} c_{33} - 2(c_{33} c_{44} - c_{66} c_{23} c_{55} - c_{12} c_{44} c_{55} + c_{13} c_{22} c_{55} - 2 c_{44} c_{55} c_{66} - c_{13} c_{44} c_{66} + c_{12} c_{33} c_{66} - c_{12} c_{13} c_{44} - c_{13} c_{23} c_{55} - c_{12} c_{23} c_{55} - c_{12} c_{13} c_{23}) - c_{13} c_{22} + c_{22} c_{33} - c_{23}] l_1^2 l_2^2 + [(2 c_{13} c_{55} - c_{66} c_{33} - c_{55} c_{44} - c_{33} - c_{66} c_{55} + c_{13}^2)] l_1^2 + (2 c_{23} c_{44} + c_{23}^2 - c_{22} c_{33} - c_{23} c_{55} - c_{33} c_{66} - c_{55} c_{44} - c_{33} c_{66}) l_2^2 + (c_{33} + c_{44} + c_{55}) \zeta^2, \]

\[
F_3 = (c_{55} l_1^2 + c_{44} l_2^2 - \zeta^2) \{[(1 + c_{66}) l_1^2 + (c_{22} + c_{66}) l_2^2] \zeta^2 - \zeta^4 + [(c_{22} c_{66} + c_{12}^2 - c_{22}) c_{55}] l_1^2 l_2^2 - c_{22} c_{66} l_2^2 - c_{66} l_1^2 \}. \]

In this case, Eqs. (14) simplify to

\[
D_1(\alpha_q) = M_{13}(\alpha_q) M_{23}(\alpha_q) - M_{12}(\alpha_q) M_{33}(\alpha_q),
\]

\[
D_2(\alpha_q) = M_{12}(\alpha_q) M_{23}(\alpha_q) - M_{13}(\alpha_q) M_{22}(\alpha_q),
\]

\[
D_3(\alpha_q) = 0,
\]

\[
D(\alpha_q) = M_{22}(\alpha_q) M_{33}(\alpha_q) - M_{23}^2(\alpha_q)
\]

and the reduced result corresponds to the purely elastic orthotropic materials, which is obtained and studied by Yamada and Nasser [29]. On the other hand, the second factor of the Eq. (26) is

\[
l_1^2 + \tilde{K}_3 l_2^2 + \tilde{K}_3 \alpha^2 - \omega^* \zeta^2 \tau = 0,
\]

which corresponds to the purely thermal wave. Hence thermal wave in the generalized theory of thermoelasticity is influenced by the thermal relaxation time \( \tau \).

4.2. Thermoelastic free waves

When layer I = II and \( h_1 = h_2 \) (say \( h \) ) then the thickness of the layer is \( 2h \), on considering origin at mid of the plate, then the above analysis reduces to a single plate. In this case, the eight roots of Eq. (12) can be arranged in four pairs as \( \alpha_{j+1} = -\alpha_j \), \( j = 1, 3, 5, 7 \).

It is observed from Eq. (11) that \( M_{13}, M_{23}, M_{34} \) and \( M_{43} \) are odd functions of \( \alpha \), and the other \( M_{ij} \)'s are even functions of \( \alpha \). On employing the thermal stresses and thermal gradient free surfaces conditions

\[
\sigma_{3j} = T' = 0, \quad x_3 = \pm h, \quad j = 1, 2, 3,
\]

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and employing the relations (14), we have
\[ \gamma_{q+1} = \gamma_q, \quad \delta_{q+1} = -\delta_q \quad \text{and} \quad \Theta_{q+1} = \Theta_q. \] (31)

Hence from (23a)
\[ b_{1q+1} = -b_{1q}, \quad b_{2q+1} = -b_{2q}, \quad b_{3q+1} = b_{3q} \quad \text{and} \quad b_{4q+1} = -b_{4q}, \] (32)
\[ b_{1j} = l_1 \delta_j - \alpha_j, \quad b_{2j} = l_2 \delta_j - \alpha_j \gamma_j, \]
\[ b_{3j} = \bar{c}_{13} l_1 + \bar{c}_{23} l_2 \gamma_j - \bar{c}_{33} \alpha_j \delta_j - \beta_j \Theta_j, \quad b_{4j} = -i \xi \alpha_j \Theta_j \bar{c}_{jk} = c_{jk}/c_{11}, \] (33)
\[ \det [P_{jk}] = 0. \] (34)

Eq. (34) is the corresponding characteristic equation for free waves in generalized thermoelasticity. Further, if thickness \( d = (h_1 + h_2) \rightarrow \infty \), in Eq. (34) then the problem reduces to thermoelastic surface waves.

4.3. Coupled thermoelasticity

This is the case, when thermal relaxation times \( \tau_0 = \tau_1 = 0 \) and hence, \( \tau = \tau_q = i/\omega \). Following above, we arrived at frequency equation of the coupled thermoelasticity. When \( \tau_1 = \tau_0 \neq 0 \), characteristic Eq. (21) becomes the frequency equation in the LS theory of generalized thermoelasticity.

5. Numerical results and discussion

Using Eq. (24a) numerical results are presented to exhibit the dependence of dispersion on the angle of propagation and thermal relaxation time. The materials chosen for this purpose are aluminum epoxy composite as layer I \( (h_1 = 0.6) \) and carbon steel as layer II \( (h_2 = 0.4) \).

Since the distinction among the wave mode types of thermoelastic waves in anisotropic plates is somewhat artificial, as the thermal and elastic wave modes are generally coupled, they are referred to as quasilongitudinal and quasitransverse, quasishear horizontal modes and quasithermal. For wave propagation in the direction of higher symmetry (see Section 4), some wave types revert to pure modes and lead to a simple characteristic equation of lower order, and consequently to the loss of pure wave modes in the direction of general propagation. Here Fig. 2 depicts the dispersion curves for the direction cosines of propagation \( l_1 = 0.259, \quad l_2 = 0.542, \quad \text{and} \quad l_3 = 0.799 \), whereas Fig. 3 demonstrate the dispersion behavior when the direction cosines of propagation are same but the coupling constant \( \varepsilon = 0 \), i.e., thermal and elastic fields are not coupled.

Similarly, dispersion curves with the direction cosines of propagation \( l_1 = 0.195, \quad l_2 = 0.515, \quad \text{and} \quad l_3 = 0.834 \) are shown in Fig. 4, whereas when the direction cosines of propagation are same but the coupling constant \( \varepsilon = 0 \).

Similarly, on considering the direction cosines of propagation \( l_1 = 0.125, \quad l_2 = 0.707, \quad \text{and} \quad l_3 = 0.696 \) dispersion curves are shown in Fig. 6, whereas when the coupling constant \( \varepsilon = 0 \), keeping the same direction cosines dispersion curves are shown in Fig. 7.

It is observed that in generalized thermoelasticity, at zero wave number limits, each figure (Figs. 2, 4 and 6) displays four thermoelastic wave speeds corresponding to one quasilongitudinal, two quasitransverse and one quasithermal. It is apparent that the largest value corresponds to the quasi-longitudinal and the additional mode appears is a quasi-thermal mode. At low wave number limits, modes are found to highly influenced and also vary with the direction. A small
change is observed in these modes values as $\xi$ increases and others higher modes appear, one of the modes seems to be associated with quick change in the slope. It is also observed that with change in direction, lower modes appear to have large influence than the higher modes where a small variation is noticed. When the when the coupling constant $\varepsilon = 0$, i.e., thermal and elastic fields are not coupled, Figs. 3, 5 and 7 demonstrate the dispersion behavior of wave speed modes with different angles of propagation. From these figures, it is observed that at low wave number limits, although wave speed modes are dispersive, but are different from the coupled case. Thus in generalized thermoelasticity, at low values of the wave number, only
Fig. 6. Phase velocity versus wave number with direction cosine \( l_1 = 0.125, \ l_2 = 0.707 \) and \( l_3 = 0.696 \) in generalized thermoelasticity

Fig. 7. Phase velocity versus wave number for the direction cosine \( l_1 = 0.125, \ l_2 = 0.707 \) and \( l_3 = 0.696 \) when the coupling parameter is zero

Fig. 8. Phase velocity versus wave number in GL theory of thermoelasticity with thermal relaxation times \( \tau_0 = 2 \cdot 10^{-7}, \ \tau_1 = 2 \cdot 10^{-6} \)

Fig. 9. Phase velocity versus wave number in GL theory of thermoelasticity with thermal relaxation times \( \tau_0 = 2 \cdot 10^{-7}, \ \tau_1 = 10 \cdot 10^{-7} \)

the lower modes highly affected and the little change is observed at the relatively high values of wave number. The low value region of the wave number is found to be of more physical interest in generalized thermoelasticity. As high wave number limits exhibit no effect on wave speeds, therefore the second sound effects are short lived in the laminated composites plates in generalized thermoelasticity.

To observe the influence of the thermal relaxations, selected values of thermal relaxation times \( \tau_1 \) and \( \tau_0 \) are considered, Figs. 8–10 demonstrate the variations of phase velocity with wave number and the dispersive character of quasilongitudinal, quasitransverse and quasither-
Fig. 10. Phase velocity versus wave number in GL theory of thermoelasticity with thermal relaxation times $\tau_0 = 2 \cdot 10^{-7}$, $\tau_1 = 4 \cdot 10^{-7}$

Fig. 11. Phase velocity versus wave number in LS theory of thermoelasticity with thermal relaxation times $\tau_0 = 2 \cdot 10^{-7}$

mal modes are represented. Quasilongitudinal, quasitransverse (two) and quasi-thermal waves are found coupled with each other due to the thermal and anisotropic effects, also wave-like behavior of the quasi-thermal modes is characterized in Green and Lindsay (GL) thermoelasticity theory. Also Fig. 11 is drawn by considering $\tau_0$ only, a single time constant which represents the dispersion curve in Lord and Shulman (LS) theory.

Although the thermal relaxation times $\tau_1$ and $\tau_0$ are derived from distinctively different physical assumptions and physical laws, the dispersion behavior described by LS and GL theory for thermoelastic waves are remarkably similar even in laminated composites plates. It is probably due to the fact that even though the theories are entirely different in their approach to form a coupled thermoelasticity theory, they are remarkably similar in their formulation.

6. Conclusion

Dispersion of a 3D layered heat conducting composite plate in an arbitrary direction in the theory of generalized thermoelasticity is studied. Equations of motion for 3D continuum formulated for an infinite layered plate of an anisotropic thermoelastic medium with uniformly distributed temperature. The Floquet method is used for the derivation of general solution of displacements and temperature distributions. Special cases such as classical, free waves and coupled thermoelasticity are also presented and discussed. Influence of wave propagation direction on plate dispersion is analysed numerically and analytically.

References
